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I.—SINGULAR SOLUTIONS AND PARTICULAR INTEGRALS OF DIFFERENTIAL EQUATIONS.

(*Continued.*)

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1. LET $c = wv^a + u$ be a regular integral of a differential equation of any order between two variables, and let $v = 0$ be a singular solution. Then differentiating with respect to x, and multiplying the result by v^{1-a}, we get

$$0 = \left(\frac{dw}{dx}\right) v + aw\left(\frac{dv}{dx}\right) + v^{1-a}\left(\frac{du}{dx}\right) = \text{V suppose.}$$

Let y_m represent the m^{th} differential coefficient of y with respect to x. Then manifestly, since a is a positive quantity less than unity, and since $\left(\frac{du}{dx}\right)$ is necessarily neither zero nor infinity for all the simultaneous values of the variables defined by the equation $v = 0$, we shall have, under the condition $v = 0$,

$\frac{dV}{dy_m} = 0$, if v involve neither y_m nor y_{m-1}.

$\frac{dV}{dy_m} = aw\frac{dv}{dy_{m-1}}$, a finite quantity, if v involve y_{m-1} and not y_m.

$\frac{dV}{dy_m} = \infty$, if v involve y_m.

Ex. Let us take the equation

$$0 = 3y_2(xy + a) + y_1(y + xy_1) + 3y_3(xy + a)^{\frac{2}{3}} = V,$$

which is derived, according to the above method, from the equation

$$c = y_1(xy + a)^{\frac{1}{3}} + y_2.$$

Then clearly, under the condition $xy + a = 0$, we have

$$\frac{dV}{dy_2} = 3(xy + a) = 0,$$

$$\frac{dV}{dy_3} = 3(xy + a)^{\frac{2}{3}} = 0,$$

$$\frac{dV}{dy_1} = y + xy_1 + xy_1 = xy_1,$$

$$\frac{dV}{dy} = 3xy_2 + xy_1^2 + 2y_3x(xy + a)^{\frac{1}{3}} = \infty,$$

$$\frac{dV}{dx} = 3yy_2 + y_1^2 + 2yy_3(xy + a)^{-\frac{1}{3}} = \infty.$$

2. Let y_r and y_s represent the r^{th} and the s^{th} differential coefficients of y. Then, differentiating the equation $V = 0$ on the hypothesis that y_r varies in consequence of the variation of y_s independently of the other variables, we get

$$\frac{dV}{dy_s} + \frac{dV}{dy_r}\frac{dy_r}{dy_s} = 0 \text{ .... } (k).$$

1st. Suppose that v involves y_s and not y_r. Then, since by the preceding article, under the condition $v = 0$, we have

$$\frac{dV}{dy_s} = \infty,$$

$$\text{and } \frac{dV}{dy_r} = \text{zero or a finite quantity.}$$

It is clear, that under these circumstances

$$\frac{dy_r}{dy_s} = \infty.$$

2nd. Suppose that v involves y_{s-1}, and none of the quantities y_s, y_{r-1}, y_r; then, since by the preceding article, under the condition $v = 0$,

$$\frac{dV}{dy_s} = \text{a finite quantity,}$$

$$\text{and } \frac{dV}{dy_r} = 0.$$

It is clear from the equation (k), that under these circumstances

$$\frac{dy_r}{dy_s} = \infty.$$

Ex. Let $c = y_3(xy + y_1)^{\frac{1}{3}} + y_2$ be a first integral of a differential equation. The differential equation derived from it, as in Art. (1), will be

$$0 = 3y_4(xy + y_1) + y_3(y + xy_1 + y_2) + 3y_3(xy + y_1)^{\frac{2}{3}}.$$

Differentiating as if y_4 and y_1 alone varied, we have

$$0 = 3\frac{dy_4}{dy_1}(xy + y_1) + 3y_4 + xy_3 + 2y_3(xy + y_1)^{-\frac{1}{3}};$$

and therefore when we put $xy + y_1 = 0$, we have evidently

$$\frac{dy_4}{dy_1} = \infty.$$

In precisely the same way we should find, under the condition $xy + y_1 = 0$, the following relations:

$$\frac{dy_4}{dx} = \infty, \quad \frac{dy_4}{dy} = \infty,$$

$$\frac{dy_3}{dx} = \infty, \quad \frac{dy_3}{dy} = \infty, \quad \frac{dy_3}{dy_1} = \infty,$$

$$\frac{dy_2}{dx} = \infty, \quad \frac{dy_2}{dy} = \infty, \quad \frac{dy_2}{dy_1} = \infty.$$

Again, to afford an illustration of the second case mentioned in this article, we will differentiate as if y_2 and y_3 alone varied, and thus we have

$$0 = \frac{dy_3}{dy_2}(y + xy_1 + y_2) + y_3 + 3\frac{dy_3}{dy_2}(xy + y_1)^{\frac{2}{3}};$$

and therefore, evidently when we put $xy + y_1 = 0$, we have

$$\frac{dy_3}{dy_2} = \infty.$$

In the same way we may shew, that under the condition $xy + y_1 = 0$,

$$\frac{dy_4}{dy_2} = \infty.$$

3. Let the equation $V = 0$ be of the n^{th} order, and the singular solution $v = 0$ of the τ^{th} order. Then clearly, under the general results of the preceding article, are comprehended the series of relations

$$\frac{dy_{\tau+1}}{dy_\tau} = \infty, \quad \frac{dy_{\tau+2}}{dy_\tau} = \infty, \quad \frac{dy_{\tau+3}}{dy_\tau} = \infty, \;\ldots\ldots\; \frac{dy_n}{dy_\tau} = \infty,$$

$$\text{and } \frac{dy_{\tau+2}}{dy_{\tau+1}} = \infty, \quad \frac{dy_{\tau+3}}{dy_{\tau+1}} = \infty, \quad \frac{dy_{\tau+4}}{dy_{\tau+1}} = \infty, \;\ldots\ldots\; \frac{dy_n}{dy_{\tau+1}} = \infty.$$

4. Let $W = 0$ represent the equation $V = 0$ under any state of

modification. Then, differentiating as in the equation (k) of Art. 2, we have

$$\frac{dW}{dy_s} + \frac{dW}{dy_r}\,\frac{dy_r}{dy_s} = 0 \;\ldots\ldots\ldots\; (g).$$

but under the circumstances of Art. (2) we have

$$\frac{dy_r}{dy_s} = \infty,$$

and therefore by equation (g) we must have

$$\frac{dW}{dy_s} : \frac{dW}{dy_r} = \infty.$$

5. Let $U = 0$ represent the equation $V = 0$, cleared of radicals. Then, by the preceding article, and under the circumstances of Art. (2), we must have

$$\frac{dU}{dy_s} : \frac{dU}{dy_r} = \infty.$$

But $\dfrac{dU}{dy_s}$ cannot assume an infinite value when submitted to the relation $v = 0$, since U, and therefore $\dfrac{dU}{dy_s}$, does not involve radicals. Hence clearly, we must have

$$\frac{dU}{dy_r} = 0.$$

Ex. Take the equation

$$0 = (1 + y_3)^2 - 4y_1^2 . (x + y_2) = U,$$

which has $x + y_2 = 0$ as a singular solution.

Then clearly, under the condition $x + y_2 = 0$, we have

$$\frac{dU}{dy} = 0,$$

$$\frac{dU}{dy_1} = -\,8y_1(x + y_2) = 0,$$

$$\frac{dU}{dy_3} = 2(1 + y_3) = 0.$$

6. Let the equation $U = 0$ be of the n^{th} order, and the singular solution $v = 0$ of the τ^{th} order. Then clearly, under the general results of the preceding article, are comprehended the relations

$$\frac{dU}{dy_{\tau+1}} = 0, \quad \frac{dU}{dy_{\tau+2}} = 0, \quad \frac{dU}{dy_{\tau+3}} = 0, \;\ldots\ldots\; \frac{dU}{dy_n} = 0,$$

7. Let $v = 0$ be a particular integral corresponding to an infinite value of c. Then putting $-\,a$ in place of a, we have

$$V = \left(\frac{dw}{dx}\right) v - aw\left(\frac{dv}{dx}\right) + v^{1+a}\left(\frac{du}{dx}\right) = 0.$$

Now the most general form for w is

$$\rho_0 + \rho_1 v^{\beta_1} + \rho_2 v^{\beta_2} + \ldots\ldots$$

where β_1, β_2, β_3, ... are all positive quantities. Hence

$$\left(\frac{dw}{dx}\right) v = \left(\frac{d\rho_0}{dx}\right) v + \left(\frac{d\rho_1}{dx}\right) v^{1+\beta_1} + \left(\frac{d\rho_2}{dx}\right) v^{1+\beta_2} + \ldots\ldots$$
$$+ (\beta_1\rho_1 v^{\beta_1} + \beta_2\rho_2 v^{\beta_2} + \ldots\ldots) \left(\frac{dv}{dx}\right);$$

and therefore we have

$$V = v^{1+a}\left(\frac{du}{dx}\right) + \left(\frac{d\rho_0}{dx}\right) v + \left(\frac{d\rho_1}{dx}\right) v^{1+\beta_1} + \left(\frac{d\rho_2}{dx}\right) v^{1+\beta_2} + \ldots$$
$$+ \left\{-a\rho_0 + (\beta_1 - a)\rho_1 v^{\beta_1} + (\beta_2 - a)\rho_2 v^{\beta_2} + \ldots\right\}\left(\frac{dv}{dx}\right) = 0 \ldots (\lambda).$$

Hence if we put $v = 0$ in the expression for $\frac{dV}{dy_m}$, we shall have

$\frac{dV}{dy_m} = 0$, if v involve neither y_m nor y_{m-1},

$\frac{dV}{dy_m} = -a\rho_0 \frac{dv}{dy_{m-1}}$, a finite quantity, if v involve y_{m-1} and not y_m,

and, if v involve y_m,

$$\frac{dV}{dy_m} = \left\{\beta_1\rho_1(\beta_1 - a) v^{\beta_1 - 1} + \beta_2\rho_2(\beta_2 - a) v^{\beta_2 - 1} + \ldots\right\} \frac{dv}{dy_m}\left(\frac{dv}{dx}\right)$$
$$+ \left(\frac{d\rho_0}{dx}\right) \frac{dv}{dy_m} - a\rho_0 \frac{d}{dy_m}\left(\frac{dv}{dx}\right)$$
$$= \left(\frac{d\rho_0}{dx}\right) \frac{dv}{dy_m} - a\rho_0 \frac{d}{dy_m}\left(\frac{dv}{dx}\right),$$

since the former part of the expression vanishes by virtue of the equation (λ).

It is also easily seen, by the aid of the equation (λ), that

$$\frac{d}{dy_m}\left(\frac{dv}{dx}\right) = 0 \text{ if } \rho_0 \text{ be a constant quantity,}$$

and that it remains finite if ρ_0 be variable. Hence

$$\frac{dV}{dy_m} = 0 \text{ if } \rho_0 \text{ be a constant quantity,}$$

and a finite quantity, if ρ_0 be variable.

Ex. Take the equation

$$0 = 2y_2(x + y_2) - y_1(1 + y_3) + 2y_1(x + y_2)^{\frac{3}{2}} = V,$$

of which $x + y_2 = 0$ is a particular integral corresponding to an infinite value of the arbitrary constant in the integral

$$c = y_1(x + y_2)^{-\frac{1}{2}} + y.$$

Then clearly, under the condition $x + y_2 = 0$, we have

$$\frac{dV}{dy_1} = -(1 + y_3) + 2(x + y_2)^{\frac{3}{2}} = 0,$$

$$\frac{dV}{dy_3} = -y_1.$$

$$\frac{dV}{dy_2} = 2(x + y_2) + 2y_2 + 3y_1(x + y_2)^{\frac{1}{2}} = 2y_2.$$

Ex. Take the equation

$$0 = -a(1 + y_3) + 2y_1(x + y_2)^{\frac{3}{2}} = V,$$

of which $x + y_2 = 0$ is a first integral corresponding to an infinite value of the arbitrary constant. Then, under the condition $x + y_2 = 0$, we have

$$\frac{dV}{dy_2} = 3y_1(x + y_2)^{\frac{1}{2}} = 0,$$

$$\frac{dV}{dx} = 3y_1(x + y_2)^{\frac{1}{2}} = 0.$$

Ex. Take the equation

$$0 = x + y_2 - x(1 + y_3) - \tfrac{1}{2}(1 + y_3)(x + y_2)^{\frac{1}{2}} + y_1(x + y_2)^2 = V,$$

of which $c = \{x + (x + y_2)^{\frac{1}{2}}\}(x + y_2)^{-1} + y$ is a regular integral, and $x + y_2 = 0$ a particular integral for $c = \infty$.

Differentiating, we have

$$\frac{dV}{dy_2} = 1 - \tfrac{1}{4}(1 + y_3)(x + y_2)^{-\frac{1}{2}} + 2y_1(x + y_2);$$

but from the equation $V = 0$, we have

$$1 + y_3 = \frac{x + y_2 + y_1(x + y_2)^2}{x + \frac{1}{2}(x + y_2)^{\frac{1}{2}}},$$

and therefore

$$\frac{dV}{dy_2} = 1 - \frac{(x + y_2)^{\frac{1}{2}} + y_1(x + y_2)^{\frac{3}{2}}}{4x + 2(x + y_2)^{\frac{1}{2}}} + 2y_1(x + y_2)$$

$$= 1 \text{ under the condition } x + y_2 = 0.$$

If we had taken the differential equation corresponding to a regular integral

$$c = \{a + (x + y_2)^{\frac{1}{2}}\}(x + y_2)^{-1} + y,$$

we should have got under the condition $x + y_2 = 0$,

$$\frac{dV}{dy_2} = 0.$$

8. Differentiating the equation $V = 0$ on the hypothesis that y_r

varies in consequence of the variation of y_s, independently of the other variables, we get

$$\frac{dV}{dy_s} + \frac{dV}{dy_r}\frac{dy_r}{dy_s} = 0 \text{ ....} \; (m).$$

1st. Suppose that v involves y_{s-1}, and none of the quantities y_s, y_{r-1}, y_r.

Then since, by the preceding article, under the condition $v = 0$,

$$\frac{dV}{dy_s} = \text{a finite quantity},$$

$$\text{and} \quad \frac{dV}{dy_r} = 0,$$

we have, by the equation (m) under these circumstances,

$$\frac{dy_r}{dy_s} = \infty.$$

2nd. Suppose that v involves y_s and y_{r-1}, and not y_r. Then since by the preceding article, under the condition $v = 0$,

$$\frac{dV}{dy_s} = \text{a finite quantity or zero},$$

$$\text{and} \quad \frac{dV}{dy_r} = \text{a finite quantity},$$

we have by the equation (m) under these circumstances,

$$\frac{dy_r}{dy_s} = \text{a finite quantity or zero}.$$

We may obviously obtain, under the condition $v=0$, for different values of r and s a variety of such relations as these.

We will furnish the following exemplifications of the two classes of relations at which we have arrived.

Ex. Take the equation

$$0 = y_3(xy + y_1) - y_2(y + xy_1 + y_2) + y_4(xy + y_1)^2,$$

of which $xy + y_1 = 0$ is a particular integral corresponding to an infinite value of the arbitrary constant of a first integral.

Differentiating as if y_2 and y_4 alone varied, we have

$$0 = -(y + xy_1 + y_2) - y_2 + \frac{dy_4}{dy_2}(xy + y_1)^2;$$

and therefore, under the condition $xy + y_1 = 0$, we clearly have

$$\frac{dy_4}{dy_2} = \infty.$$

Again, differentiating as if y and y_2 alone varied, we have

$$0 = xy_3 - (y + xy_1 + y_2)\frac{dy_2}{dy} - y_2\left(1 + \frac{dy_2}{dy}\right) + 2xy_4(xy + y_1);$$

and therefore, under the condition $xy + y_1 = 0$, we have

$$\frac{dy_2}{dy} = \frac{y_2 - xy_3}{y_2}.$$

Ex. Take the equation

$$0 = -b(y_3 + y_1) - \tfrac{1}{2}(y_3 + y_1)(y_2 + y)^{\frac{3}{2}} + y_1(y_2 + y)^2,$$

of which $c = \{b + (y_2 + y)^{\frac{1}{2}}\}(y_2 + y)^{-1} + y$ is a first integral.

Then differentiating as if y and y_3 alone varied, we have

$$0 = -b\frac{dy_3}{dy} - \tfrac{1}{2}\frac{dy_3}{dy}(y_2 + y)^{\frac{3}{2}} - 3(y_3 + y_1)(y_2 + y)^{\frac{1}{2}} + 2y_1(y_2 + y),$$

and therefore if we put $y_2 + y = 0$, we clearly have

$$\frac{dy_3}{dy} = 0.$$

9. Let the equation $V = 0$ be of the n^{th} order, and the particular integral $v = 0$ of the τ^{th} order. Then, under the condition $v = 0$, the general results of Art. (8) comprehend the following relations:

$$\frac{dy_{\tau+2}}{dy_\tau} = \infty, \quad \frac{dy_{\tau+3}}{dy_\tau} = \infty, \ldots\ldots \frac{dy_n}{dy_\tau} = \infty.$$

10. Let $U = 0$ represent the equation $V = 0$ cleared of radicals. Then differentiating on the same hypothesis as in the equation (m), we have

$$\frac{dU}{dy_s} + \frac{dU}{dy_r}\frac{dy_r}{dy_s} = 0 \ldots\ldots\ldots (w).$$

1st. Suppose that v involves y_{s-1}, and none of the three quantities y_s, y_{r-1}, y_r. Then since by Art. 8, under the condition $v = 0$, we have

$$\frac{dy_r}{dy_s} = \infty,$$

it is clear by the equation (w) that

$$\frac{dU}{dy_s} : \frac{dU}{dy_r} = \infty;$$

but $\dfrac{dU}{dy_s}$ cannot assume an infinite value, since U, and therefore $\dfrac{dU}{dy_s}$ involves no radicals. Hence clearly

$$\frac{dU}{dy_r} = 0.$$

2nd. Suppose that v involves y_s, and y_{r-1}, and not y_r. Then since by Art. 8, under the condition $v = 0$,

$$\frac{dy_r}{dy_s} = \text{a finite quantity or zero},$$

we see by the equation (w), that

$$\frac{dU}{dy_s} : \frac{dU}{dy_r} = 0, \text{ or a finite quantity,}$$

and therefore $\dfrac{dU}{dy_s} = 0$, or a finite quantity.

Ex. Take the equation

$$0 = \{a(1+y_2) - y_3(x+y_1)^2\}^2 - \tfrac{1}{4}(x+y_1)(1+y_2)^2 = U,$$

of which $c = \{a + (x+y_1)^{\frac{1}{2}}\}(x+y_1)^{-1} + y_2$ is a first integral.

Then

$$\frac{dU}{dy_3} = -2\{a(1+y_2) - y_3(x+y_1)^2\}(x+y_1)^2 = 0, \text{ putting } x+y_1 = 0;$$

also,

$$\frac{dU}{dy_1} = -4\{a(1+y_2) - y_3(x+y_1)^2\}y_3(x+y_1) - \tfrac{1}{4}(1+y_2)^2.$$

Hence, $\dfrac{dU}{dy_1} : \dfrac{dU}{dy_3} = \infty$, when we put $x + y_1 = 0$, as is readily seen when we substitute for $1 + y_2$ its value, as derived from the equation $U = 0$.

11. Let the equation $U = 0$ be of the n^{th} order, and $v = 0$ of the τ^{th}. Then, clearly, the general results of Art. 10 comprehend the following relations:

$$\frac{dU}{dy_{\tau+2}} = 0, \quad \frac{dU}{dy_{\tau+3}} = 0, \quad \frac{dU}{dy_{\tau+4}} = 0, \ldots\ldots \frac{dU}{dy_n} = 0.$$

12. If $v = 0$ had been a particular integral corresponding to a finite value of the arbitrary constant, we should have had

$$V = \left(\frac{dw}{dx}\right) v + aw\left(\frac{dv}{dx}\right) = 0,$$

instead of the expression of Art. 7. And it will be readily seen, on reverting to the analysis, that we should have arrived at precisely the same results for this as for the former expression for V.

13. Let $v = 0$ be a solution of the equation $V = 0$, and let $v = 0$ be of the τ^{th} and $V = 0$ of the n^{th} order. Suppose that $\dfrac{dy_{\tau+1}}{dy_\tau} = \infty$, under the condition $v = 0$. Then $v = 0$ cannot be a particular integral, since, as we have shewn in Art. 8, $\dfrac{dy_{\tau+1}}{dy_\tau}$ would, in this case, have been equal to zero, or to a finite quantity. Hence, manifestly the following method will enable us always to determine all the singular solutions of a differential equation of any order.

Assume

$$\frac{dy_1}{dy} = \infty, \frac{dy_2}{dy_1} = \infty, \frac{dy_3}{dy_2} = \infty, \ldots\ldots \frac{dy_n}{dy_{n-1}} = \infty,$$

and let

$$v_0 = 0, \quad v_1 = 0, \quad v_2 = 0, \ldots\ldots v_{n-1} = 0,$$

be equations which satisfy respectively the preceding relations, the subscript numbers being characteristic of their respective orders. Then, as many of these equations as satisfy the equation $V = 0$ are singular solutions.

Again, suppose that $\frac{dU}{dy_r} : \frac{dU}{dy_{r+1}} = \infty$ under the condition $v = 0$, $U = 0$ representing the equation $V = 0$, cleared of radicals. Then $v = 0$ cannot be a particular integral, since, as we have shewn in Art. 10, $\frac{dU}{dy_r} : \frac{dU}{dy_{r+1}}$ would in this case have been equal to zero or to a finite quantity. Hence, obviously, we may ascertain a singular solution of any assigned order of a differential equation by the following method :

Assume

$$\frac{dU}{dy_{r+1}} = 0, \frac{dU}{dy_{r+2}} = 0, \frac{dU}{dy_{r+3}} = 0, \ldots\ldots \frac{dU}{dy_n} = 0,$$

and eliminate $y_{r+1}, y_{r+2}, y_{r+3}, \ldots\ldots\ldots\ldots y_n$ between these equations and the equation $U = 0$. Then, if the resulting equation satisfy the relation $\frac{dU}{dy_r} : \frac{dU}{dy_{r+1}} = \infty$, it will be a singular solution.

14. Let $v = 0$ be a solution of the equation $V = 0$, and let $v = 0$ be of the r^{th} and $V = 0$ of the n^{th} order. Suppose that $\frac{dy_{r+2}}{dy_r} = \infty$, and $\frac{dy_{r+1}}{dy_r} =$ zero, or a finite quantity, under the condition $v = 0$. Then $v = 0$ cannot be a singular solution, since, as we have shewn in Art. 3, $\frac{dy_{r+1}}{dy_r}$ would in this case have been equal to infinity. Hence, by the following method, we may always ascertain all the particular integrals of every order of any differential equation.

Assume

$$\frac{dy_2}{dy} = \infty, \frac{dy_3}{dy_1} = \infty, \frac{dy_4}{dy_2} = \infty, \ldots\ldots \frac{dy_n}{dy_{n-2}} = \infty,$$

and let

$$v_0 = 0, \quad v_1 = 0, \quad v_2 = 0 \ldots\ldots v_{n-2} = 0,$$

be equations which satisfy respectively the preceding relations, the subscript numbers being characteristic of their respective orders. Then, as many of these equations as satisfy the equation $V = 0$, and dissatisfy respectively the relations

$$\frac{dy_1}{dy} = \infty, \quad \frac{dy_2}{dy_1} = \infty, \quad \frac{dy_3}{dy_2} = \infty, \ \ldots\ldots \ \frac{dy_{n-1}}{dy_{n-2}},$$

are particular integrals of the equation $V = 0$.

Again, suppose that $\frac{dU}{dy_{\tau+1}}\ \frac{dU}{dy_{\tau+2}} = \infty$, and $\frac{dU}{dy_\tau} : \frac{dU}{dy_{\tau+1}} =$ zero or a finite quantity, $U = 0$ representing the equation $V = 0$ cleared of radicals. Then $v = 0$ cannot be a singular solution, since, as we have shewn in Art. 5, $\frac{dU}{dy_\tau} : \frac{dU}{dy_{\tau+1}}$ would in this case be equal to infinity. Hence we may obviously ascertain all the particular integrals of any assigned order of which a differential equation of any order is susceptible, by the following method:

Assume

$$\frac{dU}{dy_{\tau+2}} = 0, \quad \frac{dU}{dy_{\tau+3}} = 0, \quad \frac{dU}{dy_{\tau+4}} = 0, \ \ldots\ldots \ \frac{dU}{dy_n} = 0,$$

and eliminate between these relations and the equation $U = 0$, as many of the quantities $y_{\tau+1}, y_{\tau+2}, \ldots\ldots y_n$ as possible, that is, all but one. Then, if the resulting equation be susceptible of satisfaction by an equation $v = 0$ of the τ^{th} order, $v = 0$ will be a particular integral of $U = 0$, provided that it do not make

$$\frac{dU}{dy_\tau} : \frac{dU}{dy_{\tau+1}} = \infty.$$

It is worthy of observation, that the above processes enable us to determine only those particular integrals of a differential equation, which are of an order at least two dimensions lower than the proposed equation.

15. Let $c = wv^a + u$ denote a regular integral of a differential equation, and let $P = 0$ denote the same equation cleared of radicals and of fractions. Then differentiating the equation $P = 0$ as if c varied in consequence of the variation of y_m independently of the other variables, we have

$$\frac{dP}{dy_m} + \frac{dP}{dc}\,\frac{dc}{dy_m} = 0.$$

1st. Let a be a positive quantity less than unity, and let v involve y_m. Then clearly, under the condition $v = 0$, we have

$$\frac{dc}{dy_m} = \infty,$$

and therefore

$$\frac{dP}{dy_m} : \frac{dP}{dc} = \infty,$$

for a value u of c, that is, for a value of c corresponding to a singular solution if u be a variable quantity, or a particular integral if u be constant. But evidently $\frac{dP}{dy_m}$ cannot be equal to infinity

when we put $v = 0$ or $c = u$, because P involves neither fractions nor radicals. Hence we must have for $c = u$

$$\frac{dP}{dc} = 0.$$

2nd. Let a be a negative quantity, and let v involve y_m. Then clearly, under the condition $v = 0$,

$$\frac{dc}{dy_m} = \infty,$$

and therefore

$$\frac{dP}{dy_m} : \frac{dP}{dc} = \infty,$$

for a value ∞ of c. But it is clear from the nature of P that we cannot have, as in the first case, for the value of c the relation

$$\frac{dP}{dc} = 0.$$

3rd. Let a be a positive quantity greater than or equal to unity, and let v involve y_m. Then clearly, under the condition $v = 0$,

$$\frac{dc}{dy_m} = \text{zero, or a finite quantity,}$$

and therefore $\dfrac{dP}{dy_m} : \dfrac{dP}{dc} =$ zero, or a finite quantity,

for a value of c corresponding either to no solution at all of the differential equation, u being a variable quantity, or to a particular integral, u being a constant quantity.

Hence from what has been said we are enabled to avail ourselves of the following process for the determination of the singular solutions of differential equations from their regular integrals.

Let $P = 0$ represent a regular integral cleared of radicals and of fractions, the arbitrary constant c being involved in P.

Assume $\dfrac{dP}{dc} = 0$, and let u be a variable quantity deducible from this equation as a value for c. Substitute this value of c in the expression P, and let the result be represented by P′. Then if P′ have any factor v such that, for any one of the quantities x, y, y_1, y_2, ... which v involves, when $v = 0$, and $P = 0$,

$$\frac{dP}{dy_m} : \frac{dP}{dc} = \infty,$$

$v = 0$ will be a singular solution of the differential equation belonging to $P = 0$ as a regular integral.

And if we obtain every such factor of P′, and equate each of them to zero, we shall obtain all the possible singular solutions for the value u of c.

If there be more variable values of c than one deducible from the equation $\dfrac{dP}{dc} = 0$, we must proceed in the same way with each of them; and ultimately we shall have obtained all the possible singular solutions.

Ex. Let $c = x^2 y (x+y)^{\frac{1}{3}} + x$ be the complete primitive of an equation of the first order. Clearing it of radicals, we have

$$P = (c-x)^3 - x^6 y^3 (x+y) = 0 \;\ldots\ldots\; (f),$$

$$\frac{dP}{dc} = 3(c-x)^2 = 0 \text{ suppose;}$$

$$\therefore\; c = x;$$

$$\therefore\; P' = -x^6 y^3 (x+y) = 0,$$

and the factors of P' are x, y, and $x+y$.

$$\frac{\dfrac{dP}{dx}}{\dfrac{dP}{dc}} = \frac{-3(c-x)^2 - 6x^5y^3(x+y) - x^6y^3}{3(c-x)^2},$$

$$= -1 - \tfrac{1}{3}xy \cdot \frac{5x+6y}{(x+y)^{\frac{2}{3}}}, \text{ from the equation } (f),$$

$$= -1, \text{ when we put } x = 0;$$

hence $x = 0$ is not a singular solution.

Again,
$$\frac{\dfrac{dP}{dy}}{\dfrac{dP}{dc}} = \frac{-3x^6y^2(x+y) - x^6y^3}{3(c-x)^2},$$

$$= \frac{-3x^6y^2(x+y) - x^6y^3}{3x^4y^2(x+y)^{\frac{2}{3}}},$$

$$= \frac{-3x^2(x+y) - x^2y}{3(x+y)^{\frac{2}{3}}}$$

$$= -x^{\frac{4}{3}} \text{ when we put } y = 0;$$

hence $y = 0$ is not a singular solution.

Again, it is clear that either $\dfrac{dP}{dx} : \dfrac{dP}{dc}$, or $\dfrac{dP}{dy} : \dfrac{dP}{dc}$ assumes an infinite value when we put $x + y = 0$. Hence

$$x + y = 0$$

is a singular solution of the differential equation whose complete primitive is

$$(c-x)^3 - x^6y^3(x+y) = 0.$$

If this equation had been presented to us in its original shape,

$$c = x^2y(x+y)^{\frac{1}{3}} + x,$$

of course we could have seen these conclusions at once.

It would sometimes be very troublesome to clear an equation of radicals. Thus, if

$$c = x(x+y)^{\frac{1}{3}} + y(x^2+y)^{\frac{1}{2}},$$

we should be going greatly out of our way if we were to endeavour to clear it of radicals, in order to ascertain the singular solutions of the corresponding differential equation.

II.—CASE OF AN APSIDAL ANGLE IN A PATH NEARLY CIRCULAR.

Determination of the angle between the apsides of the projection upon an horizontal plane of the path of a material particle, moving within a surface of revolution whose axis is vertical, the path of the particle being nearly circular.

Let x, y, z be the coordinates of the body's position at any time t, the axis of the surface being the axis of z. Let ρ denote the distance of the projection of the body upon the plane of xy from the axis of z, and θ the angle between ρ and x.

Then, since the forces which act upon the body have no moments about the axis of z, we have, by the principle of the conservation of areas,

$$\rho^2 \frac{d\theta}{dt} = h \text{ .... } (1),$$

where h is a constant quantity.

Again, since the velocity of the body must be the same as if it had fallen freely from its prime position to its altitude z, we have

$$\frac{d}{dt}\frac{ds^2}{dt^2} = -2g\frac{dz}{dt},$$

$$\text{but } ds^2 = d\rho^2 + \rho^2 d\theta^2 + dz^2,$$

and therefore

$$\frac{d}{dt}\left(\frac{d\rho^2}{dt^2} + \rho^2\frac{d\theta^2}{dt^2} + \frac{dz^2}{dt^2}\right) = -2g\frac{dz}{dt},$$

and therefore

$$\frac{d}{d\theta}\left\{\left(\frac{d\rho^2}{d\theta^2} + \rho^2 + \frac{dz^2}{d\rho^2}\frac{d\rho^2}{d\theta^2}\right)\frac{d\theta^2}{dt^2}\right\} = -2g\frac{dz}{d\rho}\frac{d\rho}{d\theta}.$$

Hence, from the equation (1),

$$\frac{d}{d\theta}\left\{\left(\frac{d\rho^2}{d\theta^2} + \rho^2 + \frac{dz^2}{d\rho^2}\frac{d\rho^2}{d\theta^2}\right)\frac{h^2}{\rho^4}\right\} = -2g\frac{dz}{d\rho}\frac{d\rho}{d\theta}.$$

Put $\frac{dz}{d\rho} = p$ and substitute $\frac{1}{u}$ for ρ, and we have

$$\frac{d}{d\theta}\left(\frac{du^2}{d\theta^2} + p^2\frac{du^2}{d\theta^2} + u^2\right) = \frac{2g}{h^2}\frac{p}{u^2}\frac{du}{d\theta};$$

$$\therefore \frac{d^2u}{d\theta^2}(1+p^2) + p\frac{dp}{d\theta}\frac{du}{d\theta} + u = \frac{gp}{h^2u^2} \text{ ......(2).}$$

Let $u = c + w$, c being the value of u at an apse.

Then clearly if $p = p_1$, and $\frac{dp}{d\rho} = q_1$ when $u = c$, we have

$$p = p_1 + \frac{dp_1}{dc}w \text{ nearly since } w \text{ is small,}$$

$$= p_1 - q_1\frac{w}{c^2}.$$

Hence the equation (2) becomes

$$\frac{d^2w}{d\theta^2}\left\{1 + \left(p_1 - \frac{q_1w}{c^2}\right)^2\right\} + \left(p_1 - \frac{q_1w}{c^2}\right)\frac{q_1}{c^2}\frac{dw^2}{d\theta^2} + c + w =$$

$$= \frac{g}{h^2}\left(\frac{p_1}{u^2} - \frac{q_1w}{c^2u^2}\right);$$

and therefore neglecting squares and products of small quantities, we get

$$\frac{d^2w}{dt^2}(1+p_1^2) + c + w - \frac{g}{h^2}\left(\frac{p_1}{c^2+2cw} - \frac{q_1w}{c^4}\right) = 0,$$

$$\therefore \frac{d^2w}{dt^2}(1+p_1^2) + c + w - \frac{g}{c^2h^2}\left\{p_1\left(1 - \frac{2w}{c}\right) - \frac{q_1w}{c^2}\right\} = 0;$$

$$\therefore \frac{d^2w}{dt^2}(1+p_1^2) + c + w - \frac{gp_1}{c^2h^2} + \frac{gw}{c^2h^2}\left(\frac{2p_1}{c} + \frac{q_1}{c^2}\right) = 0 \text{...(3).}$$

Now when the orbit is a circle with radius equal to $\frac{1}{c}$ we have, omitting the terms depending upon w,

$$c - \frac{gp_1}{h^2c^2} = 0,$$

$$\therefore \frac{g}{c^2h^2} = \frac{c}{p_1}.$$

Introducing this relation into the small term in the equation (3), we have

$$(1 + p_1^2)\frac{d^2w}{dt^2} + c + w - \frac{gp_1}{c^2h^2} + \frac{cw}{p_1}\left(\frac{2p_1}{c} + \frac{q_1}{c^2}\right) = 0;$$

$$\therefore \; (1 + p_1^2) \frac{d^2 w}{dt^2} + c - \frac{gp_1}{c^2 h^2} + \frac{3cp_1 + q_1}{cp_1} w = 0.$$

And from this equation, by the ordinary method, we get for the angle between the apsides, the expression

$$\pi \sqrt{\frac{cp_1 (1 + p_1^2)}{3cp_1 + q_1}},$$

$$\text{or } \pi \sqrt{\frac{p_1 (1 + p_1^2)}{3p_1 + q_1 a}}, \text{ if } c = \frac{1}{a}.$$

J. F. H.

III.—RESEARCHES ON THE THEORY OF ANALYTICAL TRANSFORMATIONS, WITH A SPECIAL APPLICATION TO THE REDUCTION OF THE GENERAL EQUATION OF THE SECOND ORDER.

By G. Boole, Waddington, near Lincoln.

Let P be a function of x and y: then it is clear that, whatever value we give to those variables, (and they are in this instance supposed to be without limitation,) P will assume some corresponding value, real or imaginary. Let us now suppose that x and y bear such relations to two other variable quantities, x' and y', that for every pair of values the former may be supposed to assume. the latter receive corresponding values. This is equivalent to supposing

$$x = f(x', y'), \quad y = f'(x', y') \; \ldots\ldots\ldots\ldots \; (1),$$

and does not in any way limit the generality which we suppose x and y to possess.

If we substitute for x and y the values supposed to be given in (1), we shall have the general equation

$$P = P';$$

and this will be true for all supposable values of x and y.

Suppose, now, $P = 0$ to be the equation of a curve, P being, as before, a function of x and y. This equation we may consider under two distinct points of view: first, as expressing a relation between x and y for each point of the curve, which is the ordinary, and I believe hitherto the only method of considering the subject; or, secondly, as expressing a particular state or condition of the function P. Geometrically speaking, this latter view is tantamount to considering any plane curve $\phi(x, y) = 0$, as

formed by the intersection of the surface $\phi(x, y) = z$ with the plane $z = 0$, that is, with the plane x, y.

Let it now be required to transform the equation $P = 0$ into the equation $P' = 0$ by the substitution of the values $x = f(x', y')$, $y = f'(x', y')$.

The order of proceeding it is here important to observe. We should first substitute in the function P the *general* values of x and y, and afterwards introduce the particular condition $P = 0$. The transformation of P into P′ is therefore independent of any relation among the variables supposed to be expressed in the condition $P = 0$. We may, therefore, by the reasoning of the preceding section, make $P = P'$, whether the values attributed to x and y satisfy the primitive equation to the curve or not. The same remark may be made respecting the various orders of differentials; we shall therefore have the following system of equations universally true:

$$P = P',$$

$$\frac{dP}{dx} = \frac{dP'}{dx} = \frac{dP'}{dx'}\frac{dx'}{dx} + \frac{dP'}{dy'}\frac{dy'}{dx},$$

$$\frac{dP}{dy} = \frac{dP'}{dy} = \frac{dP'}{dx'}\frac{dx'}{dy} + \frac{dP'}{dy'}\frac{dy'}{dy},$$

&c. &c.

In applying these principles to the transformation of any particular equation, we are at liberty, after performing the requisite differentiations, to replace the primitive condition $P = 0$ by any other which the nature of the problem may render it advisable to introduce. Interpreted into geometrical language, the above implies, that if any curve line in the plane xy be considered as formed by the intersection of the surface whose equation is

$$\phi(x, y) = z$$

with that plane; and if the co-ordinates x, y be transferred into another system x', y', not only will the line of intersection continue the same as before, but the intersecting surface also, throughout its whole extent.

In making use of the differential equations of the first or higher orders,

$$\frac{dP}{dx} = \frac{dP'}{dx}, \qquad \frac{dP}{dy} = \frac{dP'}{dy},$$

$$\frac{d^2P}{dx^2} = \frac{d^2P'}{dx^2}, \qquad \frac{d^2P}{dx\,dy} = \frac{d^2P'}{dx\,dy},$$

&c. &c.

we must be careful to introduce new conditions only after performing the differentiations, or in such a way as to produce the same result. If, for example, we have in P′ a term y'^2, and suppose $y' = \phi(x, y) = 0$, we may in the first differentiation neglect such a term, because y' will be retained as a coefficient; but we

cannot do this in taking the partial differentials of the second order which do not contain y'. Although the sum of the partial differentials of any order is zero, yet those partial differentials themselves will be susceptible of real values, which it will be necessary to take account of.

As a first application of the preceding theory, let it be required to transform the general equation of the second order for two variables,

$$Ax^2 + A'y^2 + 2Bxy + 2Cx + 2C'y + D = 0,$$

into a new equation with rectangular co-ordinates x', y', and of the form

$$A_1x'^2 + A_1'y'^2 + D_1 = 0.$$

By the reasoning of the preceding sections, the first members of these two equations must be equal for all values of x and y. As we are at liberty to introduce a new condition, and as one object to be determined is the inclination of the axes x and x', let us assume $y' = 0$; then will $x'^2 = (x - a)^2 + (y - b)^2$, making a and b the co-ordinates of the new centre.

The equation $P = P'$ gives

$$Ax^2 + A'y^2 + 2Bxy + 2Cx + 2C'y + D = A_1x'^2 + A_1'y'^2 + D_1 \text{ ... (1).}$$

The equation $\dfrac{dP}{dx} = \dfrac{dP'}{dx}$ gives, on substituting for x' its value, and making $y' = 0$ in the results,

$$Ax + By + C = A_1(x \quad a) \text{ .... (2).}$$

The equation $\dfrac{dP}{dy} = \dfrac{dP'}{dy}$ gives, moreover,

$$A'y + Bx + C' = A_1(y - b) \text{ .... (3).}$$

From the equation (2) we have, on differentiating,

$$\frac{dy}{dx} = \frac{A_1 - A}{B} = \tan x\,x';$$

and from (3), by a like process,

$$\frac{dy}{dx} = \frac{B}{A_1 - A'} = \tan x\,x';$$

equating these two expressions, we have

$$\frac{A_1 - A}{B} = \frac{B}{A_1 - A'},$$

which becomes, on reduction,

$$A_1^2 - (A + A')A_1 + AA' - B^2 = 0.$$

This equation virtually includes the system given in the former investigations, and its two roots determine A_1 and A_1'.

Had we, indeed, in lieu of the condition last named, made $x' = 0$, which would have given the equation

$$y'^2 = (x - a)^2 + (y - b)^2,$$

we should, as is evident from the symmetrical form of the equations, have obtained

$$\frac{dy}{dx} = \frac{A_1' - A}{B} = \tan xy',$$

$$\frac{dy}{dx} = \frac{B}{A_1' - A'} = \tan xy';$$

whence, by reduction,

$$A_1'^2 - (A + A')\,A_1' + AA' - B^2 = 0.$$

This equation will give the same values for A_1' as the one last obtained for A_1, and shews that the solution of either is sufficient to determine the values required.

In the equations (1), (2), and (3), assume $x = a$ and $y = b$, which we are allowed to do, since the values of a and b express the position of a point in the axis of x', for which alone these equations are true; then, observing that x' becomes equal to 0, we have

$$Aa^2 + A'b^2 + 2Bab + 2Ca + 2C'b + D = D_1 \;\ldots\ldots(4),$$

$$Aa + Bb + C = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5),$$

$$A'b + Ba + C' = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

From (5) and (6), by elimination,

$$a = \frac{BC' - A'C}{AA' - B^2}, \quad b = \frac{BC - AC'}{AA' - B^2}.$$

Multiplying (5) by a, and (6) by b, and subtracting half the sum from (4), we have

$$D_1 = Ca + C'b + D.$$

The process of differentiation employed in the solution of this problem, I propose to call *Differentiating along the New Axes.*

Ex. 2. To reduce the general equation of the second order for three variables,

$$Ax^2 + A'y^2 + A''z^2 + 2Bxy + 2B'xz + 2B''yz$$
$$+ 2Cx + 2C'y + 2C''z + D = 0,$$

to the more simple form

$$A_1x'^2 + A_1'y'^2 + A_1''z'^2 + D_1 = 0.$$

Equating these expressions, and differentiating along the axis x', by aid of the condition

$$x'^2 = (x - a)^2 + (y - b)^2 + (z - c)^2,$$

a, b, and c being co-ordinates of the new centre, we obtain

$$Ax + By + B'z + C = A_1(x - a) \;\ldots\ldots (1),$$

$$A'y + Bx + B''z + C' = A_1(y - b) \;\ldots\ldots (2),$$

$$A''z + B'x + B''y + C'' = A_1(z - c) \;\ldots\ldots (3).$$

For the new centre at which $x = a$, $y = b$, $z = c$, the above equations become

$$\mathrm{A}a + \mathrm{B}b + \mathrm{B}'c + \mathrm{C} = 0 \text{ .... } (4),$$
$$\mathrm{A}'b + \mathrm{B}a + \mathrm{B}''c + \mathrm{C}' = 0 \text{ .... } (5),$$
$$\mathrm{A}''c + \mathrm{B}'a + \mathrm{B}''b + \mathrm{C}'' = 0 \text{ .... } (6),$$

whence a, b, and c are determined; and by proceeding as in the last example,

$$\mathrm{D}_1 = \mathrm{C}a + \mathrm{C}'b + \mathrm{C}''c + \mathrm{D}.$$

If we subtract the equations (4), (5), (6) respectively from the corresponding ones (1), (2), and (3), we have, after transposing to one side,

$$(\mathrm{A} - \mathrm{A}_1)(x - a) + \mathrm{B}(y - b) + \mathrm{B}'(z - c) = 0 \text{......} (7),$$
$$(\mathrm{A}' - \mathrm{A}_1)(y - b) + \mathrm{B}(x - a) + \mathrm{B}''(z - c) = 0 \text{......} (8),$$
$$(\mathrm{A}'' - \mathrm{A}_1)(z - c) + \mathrm{B}'(x - a) + \mathrm{B}''(y - b) = 0 \text{......} (9).$$

Whence, on eliminating $x - a$, $y - b$, $z - c$, we obtain

$$\mathrm{A}_1^3 - (\mathrm{A} + \mathrm{A}' + \mathrm{A}'')\mathrm{A}_1^2 + (\mathrm{AA}' + \mathrm{AA}'' + \mathrm{A}'\mathrm{A}'' - \mathrm{B}^2 - \mathrm{B}'^2 - \mathrm{B}''^2)\mathrm{A}_1$$
$$- \mathrm{AA}'\mathrm{A}'' - 8\mathrm{BB}'\mathrm{B}'' + \mathrm{AB}''^2 + \mathrm{A}'\mathrm{B}'^2 + \mathrm{A}''\mathrm{B}^2 = 0 \text{......} (10),$$

a cubic, whose roots determine A_1, A_1', A_1''.

Finally, if from (7) and (8) we eliminate $z - c$, and from (8) and (9) $y - b$, and compare the results, we shall obtain as the symmetrical equations of the axis of x',

$$\{(\mathrm{A} - \mathrm{A}_1)\mathrm{B}'' - \mathrm{BB}'\}(x - a) = \{(\mathrm{A}' - \mathrm{A}_1)\mathrm{B}' - \mathrm{BB}''\}(y - b)$$
$$= \{(\mathrm{A}'' - \mathrm{A}_1)\mathrm{B} - \mathrm{B}'\mathrm{B}''\}(z - c).$$

In the last obtained system, it is only necessary to change A_1 into A_1' and A_1'' to exhibit the symmetrical equations of the axes of y' and z'.

In the preceding applications of theory, it has only been necessary to differentiate the even powers, x'^2, y'^2, of the new co-ordinates; an operation which is immediately effected by the aid of the characteristic equation

$$x'^2 + y'^2 + z'^2 = (x - a)^2 + (y - b)^2 + (z - c)^2.$$

When odd powers occur, it is most convenient to employ the first of the linear formulæ of transformation,

$$x' = \cos xx' . (x - a) + \cos yx' . (y - b) + \cos zx' . (z - c);$$

whence

$$\frac{dx'}{dx} = \cos xx', \quad \frac{dx'}{dy} = \cos yx', \quad \frac{dx'}{dz} = \cos zx'.$$

Ex. To reduce the general equation of the second order for three variables to the form

$$\mathrm{A}_1 x'^2 + \mathrm{A}_1' y'^2 + \mathrm{A}_1'' z'^2 + 2\mathrm{C}_1 x' = 0.$$

The solution of this problem will differ from that of the last only in consequence of the constant terms, $\mathrm{C}_1 \cos xx'$, $\mathrm{C}_1 \cos yx'$, $\mathrm{C}_1 \cos zx'$, respectively added to the second members of (1), (2), (3). Hence the cubic equation determining A_1, A_1', A_1'', and the

symmetrical equations of the axes x', y', z', will be the same as before. It will at once be seen, that the constants a, b, c, and C_1, will be determined by the system,

$$Aa + Bb + B'c + C = C_1 \cos xx',$$
$$A'b + Ba + B''c + C' = C_1 \cos yx',$$
$$A''c + B'a + B''b + C'' = C_1 \cos zx',$$
$$Aa^2 + A'b^2 + A''c^2 + 2Bab + 2B'ac + 2B''bc$$
$$+ 2Ca + 2C'b + 2C''c + D = 0.$$

The quantities $\cos xx'$, $\cos yx'$, $\cos zx'$, are known, being determined by the coefficients of the equation of the axis x'.

The last of the above equations rèduces to a simple one by the process adopted in the last example.

From the examination of (10) in the preceding example, it is apparent, that when the equation of the second order designates a paraboloid, its coefficients must satisfy the condition

$$AA'A'' + 8BB'B'' - AB''^2 - A'B'^2 - A''B^2 = 0;$$

and that the determination of A_1, A_1', A_1'', one of which, in this instance, becomes 0, will be effected by a quadratic.

In the preceding investigations we have supposed the form of the reduced equation known. In the following example I shall give an illustration of a more general method of solution, by which all the possible forms of the equation are determined, together with the general laws of the coefficients.

Assume

$$Ax^2 + A'y^2 + 2Bxy + 2Cx + 2C'y + D$$
$$= A_1x'^2 + A_1'y'^2 + 2B_1x'y' + 2C_1x' + 2C_1'y' + D_1 \ldots (A).$$

Differentiating along the axis of x' with respect to x and y, we have

$$Ax + By + C = A_1(x-a) + B_1x' \cos xy' + C_1 \cos xx' + C_1' \cos xy',$$
$$A'y + Bx + C' = A_1(y-b) + B_1x' \cos yy' + C_1 \cos yx' + C_1' \cos yy'.$$

Now $x' \cos xy' = -(y-b)$ and $x' \cos yy' = x - a$;

whence the above equations become

$$Ax + By + C = A_1(x-a) - B_1(y-b) + C_1 \cos xx' + C_1' \cos xy' \ldots (1),$$
$$A'y + Bx + C' = A_1(y-b) + B_1(x-a) + C_1 \cos yx' + C_1' \cos yy' \ldots (2).$$

Differentiating (1) and (2), we get

$$\frac{dy}{dx} = \frac{A_1 - A}{B_1 + B} = \frac{B_1 - B}{A' - A_1} = \tan xx' \ldots\ldots (3);$$

whence, on reduction,

$$A_1^2 - (A + A')A_1 + AA' - B^2 + B_1^2 = 0 \ldots\ldots (4).$$

Either of the expressions for $\frac{dy}{dx}$, given in (3), determines the

value of tan xx'. The equation (4) determines A_1 and A_1', and is evidently equivalent to the remarkable system

$$A_1 + A_1' = A + A' \quad \ldots\ldots\ldots\ldots\ldots\ldots (5),$$
$$A_1A_1' - B_1^2 = AA' - B^2 \quad \ldots\ldots\ \ldots\ldots\ldots (6).$$

In (A), and in (1) and (2), making $x = a$, $y = b$, and observing that under these suppositions x' vanishes, we have

$$Aa^2 + A'b^2 + 2Bab + 2Ca + 2C'b + D = D_1 \ldots\ldots (7),$$
$$Aa + Bb + C = C_1 \cos xx' + C_1' \cos xy' \ldots\ldots\ldots (8),$$
$$A'b + Ba + C' = C_1 \cos yx' + C_1' \cos yy' \ldots\ldots\ldots (9),$$

of which the first (7) is reducible, as in former examples, to a simple equation.

From the inspection of (3) it appears that the terms A_1, A_1', B_1, and tan xx', are connected by two necessary relations, and that we are therefore at liberty to impose two others. The quantities a, b, C_1, C_1', D_1, are connected by three equations, (7), (8), (9); here, therefore, we may impose likewise two new relations. If we assume

$$B_1 = 0, \quad C_1 = 0, \quad C_1' = 0,$$

we obtain the equations previously given for the discussion of the central system, (Ex. 1). If, on the contrary, we make

$$A_1 = 0, \quad C_1' = 0, \quad D_1 = 0,$$

we obtain a solution true for the case of the parabola.

As yet no use has been made of the higher system of equations

$$\frac{d^2P}{dx^2} = \frac{d^2P'}{dx^2}, \quad \frac{d^2P}{dx\,dy} = \frac{d^2P'}{dx\,dy}, \quad \frac{d^2P}{dy^2} = \frac{d^2P'}{dy^2}, \ \&c.$$

From these a very interesting class of solutions may be obtained. The resulting equations will generally involve a quadratic surd, and will afford a remarkable illustration of the varied and multiform combinations under which the same class of mathematical truths may be presented.

I shall here subjoin a few additional remarks and illustrations with reference to the preceding investigations. The following problem has not, so far as I am aware, been directly solved before.

Given the equations of the projections of a line of the second order on two rectangular co-ordinate planes, to determine the equations of the principal axes; together with the corresponding primitive equation of the curve.

I shall at present merely consider the case in which the equations are of the form

$$ax^2 + a'y^2 + 2bxy + c = 0 \ \ldots\ldots (1),$$
$$a_1x^2 + a_1'z^2 + 2b_1xz + c_1 = 0 \ \ldots\ldots (2).$$

The relation among the quantities x, y, z, is evidently linear,

since the curve is supposed to be coincident with a plane. Assume, therefore,

$$z = px + qy,$$

and substituting in (2), we have

$$(a_1 + a_1'p^2 + 2b_1 p)\, x^2 + (2a_1'p + 2b_1)\, qxy + a_1'q^2y^2 + c_1 = 0;$$

hence, by comparison with (1),

$$a = a_1 + a_1'p^2 + 2b_1 p, \quad b = (a_1'p + b_1)\, q, \quad a' = a_1'q^2, \quad c = c_1,$$

the solution of which gives

$$p = b\,\frac{\sqrt{a_1'} - b_1\sqrt{a'}}{a_1'\,.\sqrt{a'}}, \qquad q = \sqrt{\frac{a'}{a_1'}},$$

together with the necessary equations of condition,

$$\frac{aa' - b^2}{a'} = \frac{a_1 a_1' - b_1^2}{a_1'}, \quad c = c_1.$$

The fundamental equation

$$x^2 + y^2 + z^2 = x'^2 + y'^2$$

becomes, on substituting as before for z,

$$(1 + p^2)\, x^2 + 2pq\, xy + (1 + q^2)\, y^2 = x'^2 + y'^2 \; \ldots\ldots (3).$$

We are now prepared to apply the principle of transformation, which it is the object of this paper to develope.

Assume, therefore,

$$ax^2 + a'y^2 + 2bxy + c = \mathrm{A}x'^2 + \mathrm{A}'y'^2 + \mathrm{C} \ldots\ldots\ldots (4).$$

Differentiating along the axis of x' with respect to x and y, we obtain

$$\left.\begin{aligned} ax + by &= \mathrm{A}\,(1 + p^2)\, x + \mathrm{A}pqy \\ bx + a'y &= \mathrm{A}pqx + \mathrm{A}\,(1 + q^2)\, y \end{aligned}\right\}$$

$$\text{or}\quad \begin{aligned} &\{a - \mathrm{A}\,(1 + p^2)\}\, x + (b - \mathrm{A}pq)\, y = 0 \ldots\ldots (5), \\ &(b - \mathrm{A}pq)\, x + \{a' - \mathrm{A}\,(1 + q^2)\}\, y = 0 \ldots\ldots (6), \end{aligned}$$

whence, eliminating x and y,

$$\{a - \mathrm{A}\,(1 + p^2)\}\,\{a' - \mathrm{A}\,(1 + q^2)\} - (b - \mathrm{A}pq)^2 = 0 \;\ldots\ldots (7),$$

an equation whose roots determine A and A′.

Of equations (5) and (6), either is sufficient, when combined with the linear equation

$$z = px + qy,$$

to determine the position of the axis x'. Thus, from (6) we have

$$y = \frac{\mathrm{A}pq - b}{a' - \mathrm{A}\,(1 + q^2)} \cdot x \ldots\ldots (8);$$

and this expression, substituted in the value of z, gives

$$z = \frac{(a' - \mathrm{A})\, p - bq}{a' - \mathrm{A}\,(1 + q^2)} \cdot x.$$

whence, by comparison with (8),

$$\frac{x}{a' - A(1 + q^2)} = \frac{y}{Apq - b} = \frac{z}{(a' - A)p - bq},$$

the symmetrical equation of the axis x'. On changing A into A′, we obtain the equation of the axis y'.

Finally, on making x' and y' respectively 0 in (3) and (4), we have evidently

$$c = C.$$

The problem is therefore completely resolved.

Since the general equation of the second order may be represented under the symmetrical form

$$a(x - a)^2 + 2b(x - a)(x - \beta) + a'(y - \beta)^2 + c = 0,$$

it is evident that the solution for the more general form, both of the original and the reduced equations, may be easily derived from the above.

It is unnecessary to show that the method which has been here employed in the transformation of functions of two and three variables, is equally applicable to corresponding functions of any number of variables whatever. The transformation of equations of the third and higher orders, is likewise, on the same principle, made to depend on the solution of a *minimum* number of final equations. Neither of these cases being of any importance, I have not thought proper to extend the investigation beyond its present limits. In the case of equations of the third and higher orders, it may, however, be observed, that there would seem to exist more than one system of axes, and more than one relation among the coefficients, of the new and old equations, according to which the transformation may be effected, so as to result in a proposed form, or to satisfy given conditions.

With one or two remarks on the application of the preceding principles to the transformations required in the problem of rotation, and in the undulatory theory of light, I shall bring this communication to a close.

Let U be a function of x, y, z; then, by Maclaurin's theorem,

$$\begin{aligned} U = (U) &+ \left(\frac{dU}{dx}\right)x + \left(\frac{dU}{dy}\right)y + \left(\frac{dU}{dz}\right)z \\ &+ \frac{1}{1.2}\left\{\left(\frac{d^2U}{dx^2}\right)x^2 + \left(\frac{d^2U}{dy^2}\right)y^2 + \left(\frac{d^2U}{dz^2}\right)z^2 + 2\left(\frac{d^2U}{dx\,dy}\right)xy \right. \\ &\qquad \left. + 2\left(\frac{d^2U}{dx\,dz}\right)xz + 2\left(\frac{d^2U}{dy\,dz}\right)yz\right\} \\ &+ \&c.\ \&c. \end{aligned}$$

Imagine U to be transformed into a function of x', y', z', the equations of transformation being homogeneous and of the first degree, then

$$U = (U) + \left(\frac{dU}{dx'}\right) x' + \left(\frac{dU}{dy'}\right) y' + \left(\frac{dU}{dz'}\right) z'$$
$$+ \frac{1}{1.2}\left\{\left(\frac{d^2U}{dx'^2}\right) x'^2 + \left(\frac{d^2U}{dy'^2}\right) y'^2 + \left(\frac{d^2U}{dz'^2}\right) z'^2 + 2\left(\frac{d^2U}{dx'\,dy'}\right) x'y' + \&c.\right\}$$

Now, from the nature of the relation between x, y, z and x', y', z', the above values of U cannot be equal, unless each aggregate of homogeneous terms in the one be equivalent to the corresponding aggregate in the other. Those of the second order give, on comparison,

$$\left(\frac{d^2U}{dx^2}\right) x^2 + \left(\frac{d^2U}{dy^2}\right) y^2 + \left(\frac{d^2U}{dz^2}\right) z^2$$
$$+ 2\left(\frac{d^2U}{dx\,dy}\right) xy + 2\left(\frac{d^2U}{dx\,dy}\right) xz + 2\left(\frac{d^2U}{dy\,dz}\right) yz$$
$$= \left(\frac{d^2U}{dx'^2}\right) x'^2 + \left(\frac{d^2U}{dy'^2}\right) y'^2 + \left(\frac{d^2U}{dz'^2}\right) z'^2$$
$$+ 2\left(\frac{d^2U}{dx'\,dy'}\right) x'y' + 2\left(\frac{d^2U}{dx'\,dz'}\right) x'z' + 2\left(\frac{d^2U}{dy'\,dz'}\right) y'z'.$$

In the transformation of an equation of the second degree, it is therefore necessary that the coefficients should express the values assumed by a certain system of partial differential coefficients, when the variables vanish; and conversely, when such special values of the required system of partial differential coefficients present themselves, they may be transformed by considering them as coefficients of the equation of the second degree. The former of these cases is the one met with in the problem of rotation, the latter in the undulatory theory of light.

IV.—ON THE FAILURE OF FORMULÆ IN THE INVERSE PROCESSES OF THE DIFFERENTIAL CALCULUS.

If we apply the rule for integrating any power of x to the particular case when the index of the power is -1, we obtain a result having 0 in the denominator, and which is therefore nugatory. This is only one instance of several in which a certain relation of the subject to an inverse operation makes the general formulæ fail; and as these cases give rise to some difficulty, we shall here consider two of the most important of them. The instance to which we have alluded is so well known, that we need do no more than mention it; and for the more general case of failure when the index is of any value, the reader is referred to Art. VI. Vol. I. p. 109. The method of arriving at the true value in these cases of failure, is the same as that which we shall pursue in those we

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are about to consider. The principle is this: since the function in this particular case becomes infinite, we may so assume the arbitrary constant in the complementary function, as to make the formula for the particular value take the indeterminate form $\frac{0}{0}$, the true value of which can easily be determined by the ordinary rules. The assumption made with respect to the arbitrary constant in the complementary function, is to make it negative and infinite, so that the difference of two infinite quantities may be finite. Exactly the same principle holds in the instances we are about to consider.

Suppose we had the differential equation

$$\frac{dy}{dx} - ay = \varepsilon^{ax},$$

we should find, by the usual rule for integrating such equations,

$$y = \frac{\varepsilon^{ax}}{a - a} + C\varepsilon^{ax},$$

the form of which is nugatory. To discover the true value, let us suppose that the multiplier of y is not the same as the multiplier of x, but that the equation is

$$\frac{dy}{dx} - a_1 y = \varepsilon^{ax};$$

the integral of which is

$$y = \frac{\varepsilon^{ax}}{a - a_1} + C\varepsilon^{a_1 x}.$$

Now C being arbitrary, we may conceive it to consist of two parts, so that $C = -\frac{1}{a - a_1} + C_1$, which gives

$$y = \frac{\varepsilon^{ax} - \varepsilon^{a_1 x}}{a - a_1} + C_1 \varepsilon^{a_1 x}.$$

Now when $a_1 = a$, the first term takes the form $\frac{0}{0}$, which is indeterminate; and by the usual method its true value, when $a_1 = a$, is found to be $x\varepsilon^{ax}$, so that

$$y = x\varepsilon^{ax} + C_1 \varepsilon^{ax},$$

which is the true solution of the equation.

If the operating factor were of the order r, so that the equation was

$$\left(\frac{d}{dx} - a\right)^r y = \varepsilon^{ax},$$

we should find by the usual rule

$$y = \frac{\varepsilon^{ax}}{(a - a)^r} + (C_0 + C_1 x + \&c. + C_{r-1} x^{r-1})\, \varepsilon^{ax},$$

a nugatory result.

If we suppose the a in the operating factor to be different from the multiplier of x, we should have, by a change of the first arbitrary constant,

$$y = \frac{\varepsilon^{ax} - \varepsilon^{a_1 x}}{(a - a_1)^r} + (C_0' + C_1 x + \&c. + C_{r-1} x^{r-1}) \varepsilon^{a_1 x}.$$

If we differentiate the numerator and denominator of the first term in order to determine its value when $a_1 = a$, we find

$$y = \frac{x \varepsilon^{ax}}{r (a - a_1)^{r-1}} + (C_0' + C_1 x + \&c. + C_{r-1} x^{r-1}) \varepsilon^{a_1 x},$$

which is still nugatory when $a = a_1$. We must therefore continue the process, changing the constant in the second term of the complementary function, when we obtain

$$y = \frac{x (\varepsilon^{ax} - \varepsilon^{a_1 x})}{r (a - a_1)^{r-1}} + (C_0' + C_1' x + \&c. + C_{r-1} x^{r-1}) \varepsilon^{a_1 x},$$

the first term of which we find, as before, to be infinite when we make $a = a_1$. But by continuing the same process as before, we shall at last obtain

$$y = \frac{x^r \varepsilon^{ax}}{r (r-1) \ldots 2 . 1} + (C_0' + C_1' x + \&c. + C'_{r-1} x^{r-1}) \varepsilon^{a_1 x},$$

which, when $a_1 = a$, becomes

$$y = \frac{x^r \varepsilon^{ax}}{r (r-1) \ldots 2.1} + (C_0' + C_1' x + \&c. + C'_{r-1} x^{r-1}) \varepsilon^{ax},$$

being the true solution of the equation.

The other example which we shall here consider is particularly important, as the form of the solution occurs in the second approximation in the Lunar Theory, rendering necessary a change in the form of the equation.

It is met with in the integration of the equation

$$\frac{d^2 u}{d\theta^2} + n^2 u = \cos m\theta,$$

when $m = n$. For the general solution is

$$u = \frac{\cos m\theta}{n^2 - m^2} + C \cos n\theta + C_1 \sin n\theta,$$

the first term of which is infinite when $m = n$. But if, as before, we change the arbitrary constants in the complementary function, we can put the equation under the form

$$u = \frac{\cos m\theta - \cos n\theta}{n^2 - m^2} + C' \cos n\theta + C_1 \sin n\theta.$$

The value of the first term of this, when $m = n$, determined in the usual way, is $\dfrac{\theta \sin n\theta}{2n}$; so that

$$u = \frac{\theta \sin n\theta}{2n} + C' \cos n\theta + C_1 \sin n\theta.$$

In the same way, if the original equation were

$$\frac{d^2u}{d\theta^2} + n^2u = \sin n\theta,$$

we should find

$$u = -\frac{\theta \cos n\theta}{2n} + \mathrm{C} \cos n\theta + \mathrm{C}_1' \sin n\theta.$$

If the original equation were

$$\left(\frac{d^2}{d\theta^2} + n^2\right)^2 u = \cos m\theta,$$

we should have

$$u = \frac{\cos m\theta}{(n^2 - m^2)^2} + \left(\frac{d^2}{d\theta^2} + n^2\right)^{-1} (\mathrm{C}_0 \cos n\theta + \mathrm{C}_1 \sin n\theta);$$

which, by what we have just found, (observing that the constants are arbitrary,) is equal to

$$u = \frac{\cos m\theta}{(n^2 - m^2)^2} + \theta\,(\mathrm{C}_0 \sin n\theta - \mathrm{C}_0' \cos n\theta) + \mathrm{C}_1' \sin n\theta + \mathrm{C}_1' \cos n\theta.$$

The true value of the first term of this, when $m = n$, will be found, by the same process as in the last example, to be

$$-\frac{\theta^2 \cos n\theta}{2\,.\,1\,(2n)^2};$$

and generally, if the equation be

$$\left(\frac{d^2}{d\theta^2} + n^2\right)^r u = \cos m\theta,$$

the true value of the first term will be, when $m = n$,

$$\frac{\theta^r \cos\left(n\theta + r\dfrac{\pi}{2}\right)}{r\,(r-1)\,\ldots\ldots\,2\,.\,1\,(2n)^r}.$$

D. F. G.

V.—ON DIAMETRAL CURVES.

Let the equation to any curve of the n^{th} order be represented by

$$f(x, y) = 0 \;\ldots\ldots\ldots\ldots\ldots\; (1).$$

Let the equation to any straight line cutting any number of the branches of the curve be

$$y = mx + p \;\ldots\ldots\ldots\ldots\; (2).$$

Let α, β be the co-ordinates of the middle point of any portion of this straight line, intercepted between any two of the branches of the curve.

Transfer the origin of co-ordinates to the point (α, β). Then the equation (1) will become

$$f(\alpha + x', \beta + y') = 0 \text{ .... } (3),$$

and the equation (2) will become

$$\beta + y' = m(\alpha + x') + p;$$

or, since evidently

$$\beta = m\alpha + p,$$

it will become

$$y' = mx' \text{ .... } (4).$$

Combining (3) and (4), we have the following equation in x' for the intersection of the straight line and the curve,

$$f(\alpha + x', \beta + mx') = 0,$$

or $f(\alpha, \beta) + x'\left(\frac{d}{d\alpha} + m\frac{d}{d\beta}\right)f(\alpha, \beta) + \frac{x'^2}{1.2}\left(\frac{d}{d\alpha} + m\frac{d}{d\beta}\right)^2 f(\alpha, \beta)$

$$+ \frac{x'^3}{1.2.3}\left(\frac{d}{d\alpha} + m\frac{d}{d\beta}\right)^3 f(\alpha, \beta) + \text{\&c.}$$

$$+ \frac{x'^n}{1.2.3 \ldots n}\left(\frac{d}{d\alpha} + m\frac{d}{d\beta}\right)^n f(\alpha, \beta) = 0.$$

Hence, putting

$$f(\alpha, \beta) = k_0,$$

$$\left(\frac{d}{d\alpha} + m\frac{d}{d\beta}\right)f(\alpha, \beta) = k_1,$$

$$\frac{1}{1 \cdot 2}\left(\frac{d}{d\alpha} + m\frac{d}{d\beta}\right)^2 f(\alpha, \beta) = k_2,$$

$$\text{\&c.} = \text{\&c.}$$

$$\frac{1}{1 \cdot 2 \cdot 3 \ldots n}\left(\frac{d}{d\alpha} + m\frac{d}{d\beta}\right)^n f(\alpha, \beta) = k_n,$$

our equation becomes

$$k_0 + k_1x' + k_2x'^2 + k_3x'^3 + \ldots + k_nx'^n = 0.$$

Now, from the peculiar position of the new origin of co-ordinates, it is clear that this equation must contain two equal roots with opposite signs, and therefore we may represent the roots of the equation by

$$\rho_1, \rho_2, \rho_3 \ldots\ldots \rho_{n-2}, a, -a.$$

Let S_1 = sum of $-\rho_1, -\rho_2, -\rho_3, \ldots -\rho_{n-2}, -a, a$,

S_2 = sum of the products of these quantities taken two together,

S_3 = ditto taken three together,

&c. = &c.

S_n = ditto taken n together.

Then, by the theory of equations, we have the following relations:

$$k_0 = S_n k_n,$$
$$k_1 = S_{n-1} k_n,$$
$$k_2 = S_{n-2} k_n,$$
$$\&c. = \&c.$$
$$k_{n-1} = S_1 . k_n;$$

and therefore, if between these n equations we eliminate the $n - 1$ quantities $\rho_1, \rho_2, \rho_3, \ldots\ldots \rho_{n-2}, a$, we shall get an equation

$$\phi(a, \beta) = 0 \;\ldots\ldots\ldots\ldots (5),$$

which will be the equation to the diametral curve.

It is evident that the order of the equation (5) will be designated by the formula $\frac{n(n-1)}{1.2}$, which expresses the number of different pairs of intersections which the straight line (2) experiences from the different branches of the curve (1).

Ex. To determine the equation to the diametral curve of the curve

$$y^3 = bx,$$

we have

$$S_n = (-\rho_1).(-a).(a) = a^2\rho_1,$$
$$S_{n-1} = (-a).(-\rho_1) + a(-\rho_1) + (-a).(a) = -a^2,$$

and

$$S_{n-2} = a - a - \rho_1 = -\rho_1;$$
$$\therefore S_n = S_{n-1} . S_{n-2};$$
$$\therefore k_0 . k_n = k_1 . k_2,$$

or, since $n = 3$,

$$k_0 . k_3 = k_1 . k_2.$$

But $\quad k_0 = f(a, \beta) = \beta^3 - ba,$

$$k_1 = \left(\frac{d}{da} + m\frac{d}{d\beta}\right) f(a, \beta) = 3m\beta^2 - b,$$

$$k_2 = \frac{1}{1.2}\left(\frac{d}{da} + m\frac{d}{d\beta}\right)^2 f(a, \beta) = 3m^2\beta,$$

and $\quad k_3 = \frac{1}{1.2.3}\left(\frac{d}{da} + m\frac{d}{d\beta}\right)^3 f(a, \beta) = m^3.$

Hence, the required equation is

$$(\beta^3 - ba)\, m^3 = (3m\beta^2 - b) . 3m^2\beta,$$
$$\therefore m(\beta^3 - ba) = 3\beta(3m\beta^2 - b);$$
$$\therefore 8m\beta^3 - 3b\beta + mba = 0.$$

W. W.

VI.—ON A CERTAIN PROPERTY IN THE THEORY OF NUMBERS.*

Mr. Barlow, in his *Theory of Numbers*, has treated of the properties of the equation

$$x^m - b = \mathrm{M}\,.\,(n),$$

the letter M being used as an abbreviation of the words "multiple of." This is perhaps more familiar to many persons under the form

$$\frac{x^m - b}{n} = \text{an integer.}$$

In what follows it is proposed to extend some of the properties of this equation to the more general one,

$$\mathrm{A}_0 x^m + \mathrm{A}_1 x^{m-1} + \mathrm{A}_2 x^{m-2} + \ldots + \mathrm{A}_{m-1} x + \mathrm{A}_m = \mathrm{M}\,.\,(n),$$

n being a prime number, and the quantities A_0, A_1, A_2 ... A_m being integers, of which A_0 is supposed prime to n. The integer values of x *less than* n, which satisfy this equation, I call *primary* roots; those which are $> n$ but $< 2n$, *secondary* roots, and so on.

1. The equation

$$\mathrm{A}_0 x^m + \mathrm{A}_1 x^{m-1} + \ldots + \mathrm{A}_m = \mathrm{M}\,.\,(n),$$

cannot have more than m primary roots. For, if possible, let it have $m + 1$ primary roots, viz. r_1, r_2, r_3 ... r_{m+1};

$$\therefore\ \mathrm{A}_0 r_1^{\,m} + \mathrm{A}_1 r_1^{\,m-1} + \ldots + \mathrm{A}_{m-1} r_1 + \mathrm{A}_m = \mathrm{M}\,.\,(n),$$

$$\text{and}\ \mathrm{A}_0 r_2^{\,m} + \mathrm{A}_1 r_2^{\,m-1} + \ldots + \mathrm{A}_{m-1} r_2 + \mathrm{A}_m = \mathrm{M}\,.\,(n).$$

Subtracting the latter of these from the former, we have

$$\mathrm{A}_0 (r_1^{\,m} - r_2^{\,m}) + \mathrm{A}_1 (r_1^{\,m-1} - r_2^{\,m-2}) + \ldots + \mathrm{A}_{m-1} (r_1 - r_2) = \mathrm{M}\,.\,(n).$$

Now, $r_1 - r_2$ is $< n$, and therefore prime to n; we may consequently divide by it.

Hence,

$$\mathrm{A}_0 (r_1^{\,m-1} + r_2 r_1^{\,m-2} + \ldots + r_2^{\,m-1}) + \ldots + \mathrm{A}_{m-2} (r_1 + r_2) + \mathrm{A}_{m-1} = \mathrm{M}\,.\,(n).$$

Had we used r_3 instead of r_2, we should have had

$$\mathrm{A}_0 (r_1^{\,m-1} + r_3 r_1^{\,m-2} + \ldots + r_3^{\,m-1}) + \ldots + \mathrm{A}_{m-2} (r_1 + r_3) + \mathrm{A}_{m-1} = \mathrm{M}\,.\,(n).$$

Subtracting this from the preceding, and dividing by $r_2 - r_3$, we find

$$\mathrm{A}_0 \{ r_1^{\,m-2} + (r_2 + r_3)\,.\,r^{m-3} + \ldots + r_2^{\,m-2} + \ldots + r_3^{\,m-2} \} + \ldots + \mathrm{A}_{m-2} = \mathrm{M}\,.\,(n).$$

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* From a Correspondent.

Writing r_4 for r_3, subtracting and dividing as before, we shall reduce the equation to the dimensions $m-3$ in r, and lose the coefficient A_{m-2} from the end. It is sufficiently obvious, that by pursuing this process as far as to the substitution of r_m, and then dividing by $r_{m-1}-r_m$, we shall have lost successively the coefficients A_m, A_{m-1}, ... A_2, and the equation will be of one dimension in r_1; and as the term, of which A_0 is the coefficient, involves symmetrically all the quantities substituted, it must therefore be the following,

$$A_0(r_1+r_2+r_3+\dots+r_m)+A_1=M.(n).$$

Had we employed r_{m+1} *instead* of r_m, we should have found

$$A_0(r_1+r_2+\dots+r_{m-1}+r_{m+1})+A_1=M.(n);$$

$$\therefore\ A_0(r_m-r_{m+1})=M.(n).$$

But this is absurd, for both A_0 and (r_m-r_{m+1}) are prime to n, by hypothesis; and consequently the proposed equation cannot have so many as $m+1$ primary roots.

Cor. If we have an equation

$$A_0x^m+A_1x^{m-1}+\dots+A_{m-1}x+A_m=M.(n),$$

which we know to have $m+1$ primary roots, A_0 must be divisible by n, since, as before,

$$A_0(r_m-r_{m+1})=M.(n).$$

But in this case the equation reduces itself to

$$A_1x^{m-1}+A_2x^{m-2}+\dots+A_{m-1}x+A_m=M.(n);$$

for the first term, A_0x^m, being always a multiple of n, may be rejected. This new equation having the $m+1$ primary roots of the original one, we must have A_1 a multiple of n; and, by continuing this reasoning, it will follow that all the coefficients A_0, A_1, A_2, ... A_m, are multiples of n.

2. When the equation

$$A_0x^m+A_1x^{m-1}+\dots+A_{m-1}x+A_m=M.(n),$$

has *exactly* m primary roots (A_0 being prime to n), and S_r be used to denote the sum of the products of those roots taken r and r together, then the quantities

$$A_0S_1+A_1,\quad A_0S_2-A_2,\quad A_0S_3-A_3,\ \dots\ A_0S_m+(-1)^{m+1}A_m,$$

are all multiples of n.

Let x denote any one of the primary roots r_1, r_2, r_3, ... r_m; then it is evident that

$$\begin{aligned}0&=A_0(x-r_1)(x-r_2)\dots(x-r_m),\\&=A_0x^m-A_0S_1x^{m-1}+A_0S_2x^{m-2}-\dots\pm A_0S_m.\end{aligned}$$

But $M.(n)=A_0x^m+A_1x^{m-1}+A_2x^{m-2}+\dots+A_m$.

Hence, taking the difference of these two equations, we find

$$M.(n)=(A_0S_1+A_1)x^{m-1}-(A_0S_2-A_2)x^{m-2}+\dots\mp(A_0S_m\mp A_m).$$

Now x denotes *any* one of the quantities r_1, r_2, ... r_m, and consequently this equation, which is of $m - 1$ dimensions, has m primary roots; therefore, by the last corollary, its coefficients are multiples of n.

3. It seldom happens that an equation of the form we have proposed has as many primary roots as dimensions: there is, however, one which fulfils this condition, and is familiar to most readers,—it is

$$x^{n-1} - 1 = \mathrm{M}.(n),$$

which constitutes Fermat's theorem.

In this particular equation $A_0 = 1$, $A_1 = A_2 = A_3 = \ldots = A_{m-1} = 0$, and $A_m = -1$: and the primary roots are 1, 2, 3, ... $(n - 1)$. Hence

$$S_1 = 1 + 2 + 3 + \ldots + (n - 1),$$
$$S_2 = 1.2 + 1.3 + 2.3 + \ldots + (n - 2)(n - 1),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$S_{n-2} = 1.2.3 \ldots (n - 1)\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots \frac{1}{n - 1}\right),$$
$$S_{n-1} = 1.2.3 \ldots (n - 1).$$

And, by the general properties proved above, the quantities S_1, S_2, S_3, ... S_{n-2}, are all multiples of n: also, since $A_m = -1$, we have corresponding to the term $A_0 S_m - A_m$, the quantity

$$\{1.2.3 \ldots (n - 1)\} + 1,$$

a multiple of n, which is Wilson's theorem.

This is a complete solution of one of the questions proposed in the *Cambridge Problems* for 1836.

V.

VII.—ON FRESNEL'S METHOD OF FINDING, BY APPROXIMATION, THE VALUES OF THE DEFINITE INTEGRALS,

$$\int_a^b dx \,.\, {\sin \atop \cos}\{\pi . f(x)\},$$

WHERE $f(x)$ IS ANY FUNCTION OF x EXPRESSIBLE IN A SERIES OF A FINITE NUMBER OF TERMS.*

THE method employed by Fresnel for obtaining the two integrals,†

$$\int dx \,.\, \sin\left(\frac{\pi}{2}x^2\right) \text{ and } \int dx \,.\, \cos\left(\frac{\pi}{2}x^2\right),$$

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* From a Correspondent.

† See Memoires de L' Institut for 1821 and 1822, p. 407.

between any given limits, is so simple and apparently so obvious, that it would seem scarcely necessary to discuss it here, if we had not known a recent instance of an integral of this form puzzling skilful analysts for several years, and which at length was only computed by an extraordinary expenditure of labour.

Fresnel lays no claim to novelty in the method, but until it is shewn that it had been discovered by some one else previously, we must attribute the full merit of it to his fertile genius. The fact of its being overlooked shews how slightly Fresnel's last paper on diffraction has been studied by those who felt justified in speaking positively on its contents, and with whom an integration between the limits, x equal to a small quantity and x equal to infinity, was accepted as an adequate approximation to an integration, which the experiments in reality required, between the limits x equal to a small quantity and x equal to about 180. To have given the arguments built upon this case the slightest legitimate weight, this substitution ought to have been discussed.

The problem under discussion was the diffraction by a single straight edge of an opaque plate; and the experiments in fact were made with an aperture with parallel straight edges, having only the distance of a centimetre, or about 4-10ths of an inch. The second edge of such an aperture does not, doubtless, sensibly affect the diffraction by the other edge at such distances as were used: but when the arguments are drawn from very small differences, as in this case, it is clear that it should be shewn from theory as well as from experiments, that such an aperture might be substituted for a single edge. Fresnel says, p. 429, "J' avois soin que les plaques fussent séparées par un intervalle assez grand pour que l' une n' eût aucune influence sur les franges produites par l' autre. Dans presque toutes mes observations, cet intervalle était d' un centimètre."

First, let the integral $\int dx \,.\cos\{\pi . f(x)\}$ be taken in its most general form,

$$\int dx \,.\cos\{\pi(a + bx^{\alpha} + cx^{\beta} + dx^{\gamma} + \&c.)\},$$

where α, β, γ are integral or fractional, positive or negative.

Let now $x = m + v$, where m is any constant, and v a variable quantity, which is always very small: then

$$\begin{aligned}
&\int dx \,.\cos\{\pi . f(x)\} \\
&\quad = \int dv \,.\cos\{\pi\,[a + b\,(m+v)^{\alpha} + c.(m+v)^{\beta} + d.(m+v)^{\gamma} + \&c.]\} \\
&\quad = \int dv \,.\cos\,[\pi\,\{a + bm^{\alpha} + cm^{\beta} + dm^{\gamma} + \&c. \\
&\qquad\qquad + v\,.\,(b\alpha m^{\alpha-1} + c\beta m^{\beta-1} + d\gamma m^{\gamma-1}) \\
&\qquad\qquad + \text{terms containing } v^2,\ v^3,\ \&c.\}\,]
\end{aligned}$$

Since v is very small, the terms involving v^2, v^3, &c. will not sensibly increase or diminish the sum of the constant term plus that involving the first power of v;

$$\therefore \int dx \,.\cos\{\pi\,.f(x)\}$$
$$=\int dv\,.\cos\{\pi\,[a+bm^{\alpha}+cm^{\beta}+dm^{\gamma}+\&c. + v\,.\,(b\alpha m^{\alpha-1}+c\beta m^{\beta-1}+d\gamma m^{\gamma-1}+\&c.)]\}$$
$$=\frac{1}{\pi\,(b\alpha m^{\alpha-1}+c\beta m^{\beta-1}+d\gamma m^{\gamma-1}+\&c.)}$$
$$\times\sin\{\pi\,[a+bm^{\alpha}+cm^{\beta}+dm^{\gamma}+\&c. + v\,.\,(b\alpha m^{\alpha-1}+c\beta m^{\beta-1}+d\gamma m^{\gamma-1}+\&c.)]\}+\mathrm{C}.$$

Let the integral be taken between the limits $x=m$ and $x=m+h$, or between $v=0$ and $v=h$, then

$$\int_m^{m+h} dx\,.\cos\{\pi\,.f(x)\}=\frac{1}{\pi\,(b\alpha m^{\alpha-1}+c\beta m^{\beta-1}+\&c.)}$$
$$\times\left(\begin{array}{r}\sin\{\pi\,[a+bm^{\alpha}+cm^{\beta}+\&c.\ldots+h\,.\,(b\alpha m^{\alpha-1}+c\beta m^{\beta-1}+\&c.)]\}\\ -\sin\{\pi(a+bm^{\alpha}+cm^{\beta}+\&c.)\}\end{array}\right)$$
$$=\frac{2}{\pi\,(b\alpha m^{\alpha-1}+c\beta m^{\beta-1}+\&c.)}$$
$$\times\cos\{\pi\,[a+bm^{\alpha}+cm^{\beta}+\&c.\ldots+\frac{h}{2}(b\alpha m^{\alpha-1}+c\beta m^{\beta-1}+\&c.)]\}$$
$$\times\sin\left\{\frac{\pi h}{2}\,(b\alpha m^{\alpha-1}+c\beta m^{\beta-1}+d\gamma m^{\gamma-1}+\&c.)\right\}$$

In the same way we have

$$\int dx\,.\sin\{\pi\,.f(x)\}=\mathrm{C}-\frac{1}{\pi\,(b\alpha m^{\alpha-1}+c\beta m^{\beta-1}+\&c.)}$$
$$\times\cos\{\pi\,[a+bm^{\alpha}+cm^{\beta}+\&c.+v\,.\,(b\alpha m^{\alpha-1}+c\beta m^{\beta-1}+\&c.)]\};$$

and between the limits $x=m$ and $x=m+h$,

$$\int_m^{m+h} dx\,.\sin\{\pi\,.f(x)\}=\frac{2}{\pi\,(b\alpha m^{\alpha-1}+c\beta m^{\beta-1}+\&c.)}$$
$$\times\sin\{\pi\,[a+bm^{\alpha}+cm^{\beta}+\&c.+\frac{h}{2}(b\alpha m^{\alpha-1}+c\beta m^{\beta-1}+\&c.)]\}$$
$$\times\sin\left\{\frac{\pi h}{2}\,(b\alpha m^{\alpha-1}+c\beta m^{\beta-1}+\&c.)\right\}$$

To apply these formulæ to particular cases, we shall first take Fresnel's integrals,

$$\int dx\,.\cos\left(\frac{\pi}{2}\,x^2\right)\ \text{and}\ \int dx\,.\sin\left(\frac{\pi}{2}\,x^2\right),$$

and we have

$$\begin{aligned} a&=0,\\ b&=\tfrac{1}{2}\ \ldots\ \alpha=2,\\ c&=0,\\ d&=0,\\ &\&c.\end{aligned}$$

$$\therefore \int^{m+h} dx \,.\, \cos\left(\frac{\pi}{2}x^2\right) = \int^{m} dx \,.\, \cos\left(\frac{\pi}{2}x^2\right)$$

$$+ \frac{2}{\pi m} \cos\left\{\frac{\pi m}{2}(m+h)\right\} \sin\left(\frac{\pi h}{2}m\right),$$

$$\int^{m+h} dx \,.\, \sin\left(\frac{\pi}{2}x^2\right) = \int^{m} dx \,.\, \cos\left(\frac{\pi}{2}x^2\right)$$

$$+ \frac{2}{\pi m} \sin\left\{\frac{\pi}{2}m(m+h)\right\} \sin\left(\frac{\pi h}{2}m\right).$$

Bernoulli's series is convergent for these integrals when m is small, but becomes divergent when m is large; so that when we have calculated the values of the integrals for small values of x by Bernoulli's series, we can continue the computations for larger and larger values without limit, by the above simple formulæ.

For another example, we shall take the Astronomer Royal's integral in his Paper on the Intensity of Light near a Caustic, in the *Cambridge Phil. Trans.* for 1838, namely,

$$\int dx \,.\, \cos\left\{\frac{\pi}{2}(x^3 - ax)\right\}.$$

Comparing with our general solution, we have

$$a = 0,$$

$$b = \frac{1}{2}, \qquad \alpha = 3,$$

$$c = -\frac{a}{2}, \quad \beta = 1,$$

and

$$\int^{m+h} dx \,.\, \cos\left\{\frac{\pi}{2}(x^3 - ax)\right\} = \int^{m} dx \,.\, \cos\left\{\frac{\pi}{2}(x^3 - ax)\right\}$$

$$+ \frac{4}{\pi(3m^2 - a)} \cos\left\{\frac{\pi}{2}\left(m^3 - am + \frac{h}{2}(3m^2 - a)\right)\right\}$$

$$\times \sin\left\{\frac{\pi h}{4}(3m^2 - a)\right\}.$$

In the same manner as with the former integrals, this formula would be used when the value of m became so large as to cause Bernoulli's series to be divergent.

R.

VIII.—ON THE GENERAL THEORY OF THE LOCI OF CURVILINEAR INTERSECTION.

1. THE algebraical equation which comprehends the locus of the intersections of two series of curves, connected together by some assigned law, results in all cases from the elimination of an arbitrary quantity a between two equations,

$$f(x, y, a) = 0 \ldots\ldots\ldots\ldots\ldots (1),$$

$$\text{and } \phi(x, y, a) = 0 \ldots\ldots\ldots\ldots\ldots (2).$$

In the original conception of the problem, the symbol a designates some specific quantity, and the equations (1) and (2) represent two specific curves. But in the equation which results from the elimination of a between them, it is evident that all traces are lost of the peculiar meaning which may originally have been attached to it, and that consequently the final equation in x and y must comprehend and represent, not only the peculiar locus for which the investigation may have been particularly instituted, but likewise any other which any variation of the conceived affections of a may render appropriate. It is also equally manifest, that whatever curve the final equation may comprehend, there must always exist geometrical meanings for the equations (1) and (2), from which it has arisen. The truth of this converse proposition will be immediately acknowledged when we recollect, that according to the general theory of algebraical affections every geometrical equation in x and y is significant.

2. We will proceed to apply the above principles to the consideration of a class of loci which belong to the intersection of tangents at any points in the general branches of a curve with straight lines, connected with these tangents by some assigned law, and passing through some assigned point.

Let the equation to the curve, cleared of radicals, be

$$f(x, y) = 0 \ldots\ldots\ldots\ldots (3),$$

and to the tangent line at any point

$$y = \alpha x + \beta \ldots\ldots\ldots\ldots (4).$$

At the intersection of the curves (3) and (4), we have

$$f(x, \alpha x + \beta) = 0 \ldots\ldots (5);$$

and since from the nature of the contact this equation must have two equal roots, it is clear that the elimination of x between the two following equations

$$f(x, \alpha x + \beta) = 0,$$

$$\text{and } \frac{d}{dx} f(x, \alpha x + \beta) = 0,$$

must give rise to an equation

$$\phi(\alpha, \beta) = 0 \ldots\ldots\ldots\ldots (6),$$

which expresses all the appropriate relations between a and β, the dimensions of this equation in β corresponding to the number of parallel curvilinear elements which exist at different points of the curve (3).

Suppose, now, that from (6) we determine a value $\theta(a)$ for β, where $\theta(a)$ denotes some function of a, and we shall have for the equation to a tangent line

$$y = ax + \theta(a) \ldots\ldots\ldots (7).$$

Again, let the equation to a line passing through a point a, b, and connected by some law with the line (7), be represented by

$$y - b = \chi(a)(x - a) \ldots\ldots (8).$$

Then manifestly, if between (7) and (8) we eliminate a, we shall obtain an equation in x and y comprehending all the loci belonging to the infinite number of values of a, as expressed under the general form $+^r\mu$, where r is any number whatever, and μ a symbol of quantity.

It is easy to see, on a little reflection, that if we eliminate β any how between the three equations (4), (6), and (8), instead of pursuing the process above described, we shall obtain a final equation in x and y which will comprehend all the loci we have just been considering, which belong to every solution of the equation (6) for β in terms of a.

3. Suppose that, to take a particular form for the symbol of functionality χ, we assume

$$\chi(a) = -\frac{1}{a};$$

then the equation (8) will become

$$y - b = -\frac{1}{a}(x - a) \ldots\ldots\ldots (9),$$

and therefore from (7) we have for the locus of the intersection for the value $\theta(a)$ of β, the equation

$$y = -\frac{x(x-a)}{y-b} + \theta\left(-\frac{x-a}{y-b}\right) \ldots\ldots (10).$$

4. If we take $a = (-)^{\frac{1}{2}}\mu$, then the equation (9) becomes

$$y - b = -\frac{1}{(-)^{\frac{1}{2}}\mu}(x - a),$$

$$\text{or } y - b = (-)^{\frac{1}{2}}\frac{1}{\mu}(x - a);$$

and consequently, since for all values of a we must get the same equation (10), it appears that the same final equation must arise for the expression of the loci of the intersections of the curves

$$y - b = (-)^{\frac{1}{2}}\frac{1}{\mu}(x - a),$$

$$\text{and } y = (-)^{\frac{1}{2}}\mu x + \theta\,\{(-)^{\frac{1}{2}}\mu\};$$

as for that of the loci of the intersections of

$$y - b = -\frac{1}{\mu}(x - a),$$

$$\text{and } y = \mu x + \theta(\mu).$$

5. The general conclusions of the preceding article are susceptible of elegant illustrations in the conic sections.

First, let us take the parabola

$$y^2 = 4mx \text{ .... } (1).$$

Then the equation to a tangent to it in the plane $(+, +)$ will be

$$y = \mu x + \frac{m}{\mu} \text{ .... } (2),$$

and to a perpendicular upon it through the focus

$$y = -\frac{1}{\mu}(x - m) \text{ ..... } (3).$$

Now, from Art. 4, we know that the equation resulting from the elimination of μ between these two equations, will be the same as between the equations

$$y = (-)^{\frac{1}{2}}\mu x + \frac{m}{(-)^{\frac{1}{2}}\mu} \text{ .... } (4),$$

$$\text{and } y = (-)^{\frac{1}{2}}\frac{1}{\mu}(x - m) \text{ .... } (5).$$

But the equation (4) is that of a tangent to the parabola (1) in the plane $(+, +^{\frac{1}{4}})$, as will easily be ascertained on determining the value of β, from the consideration that the equation

$$\{(-)^{\frac{1}{2}}\mu x + \beta\}^2 = 4mx$$

has two equal roots.

And the equation (5) represents a straight line in the same plane with (4), which makes angles with the axis of x *complementary* to those made with it by (4).

Hence the algebraical expression of the solution of the problem, "To find the equation to the locus of the intersections of tangents to a parabola in the plane $(+, +)$, with perpendiculars upon them from the focus," will comprehend also the solution of the problem, "To find the equation to the locus of the intersections of tangents to the branch of the parabola in the plane $(+, +^{\frac{1}{4}})$, with straight lines passing through the focus of the branch in the plane $(+, +)$, and making with the axis of x angles *complementary* to those which are made with it by the tangents."

We will accordingly proceed to eliminate μ between the equations (2) and (3), and subtracting (3) from (2), we have

$$0 = \left(\mu + \frac{1}{\mu}\right) x,$$

$$\text{and therefore } x\left(\frac{x-m}{y} + \frac{y}{x-m}\right) = 0.$$

The solution

$$x = 0$$

belongs to the former problem, and represents the axis of y, while the solution

$$\frac{x-m}{y} + \frac{y}{x-m} = 0,$$

$$\text{or } y^2 + (x-m)^2 = 0,$$

belongs to the latter problem, and designates two straight lines in the plane $(+, +^{\frac{1}{4}})$, passing through the focus of the branch in the plane $(+, +)$, and inclined on each side at angles of 45° to the axis of x.

That they *should* be straight lines, is evident from the solution of the following problem in the plane $(+, +)$: "To find the equation to the locus of the intersections of tangents to a parabola with straight lines passing through the foot of the directrix, and making angles with the axis of x complementary to those made by the respective tangents."

The equation to the locus will evidently result from the elimination of μ between the two following equations:

$$y = \mu x + \frac{m}{\mu},$$

$$\text{and } y = \frac{1}{\mu}(x + m).$$

By subtraction we have

$$0 = \left(\mu - \frac{1}{\mu}\right) x,$$

$$\text{and therefore } 0 = \left(\frac{x+m}{y} - \frac{y}{x+m}\right) x,$$

$$\text{and therefore } y^2 = (x+m)^2,$$

which represent two straight lines in the plane $+, +$. The solution

$$x = 0,$$

on the contrary, denotes the locus of the intersections of perpendiculars from the focus of the parabolic branch in the plane $(+, +^{\frac{1}{4}})$, with tangents to this branch, being the axis of y in the plane $(+, +^{\frac{1}{4}})$.

We will next take the ellipse, whose equation is

$$a^2y^2 + b^2x^2 = a^2b^2 \text{.........(1).}$$

The equations in this case corresponding to the equations (2) and (3) in the preceding investigations for the parabola, will be respectively

$$y = \mu x + \sqrt{a^2\mu^2 + b^2} \text{ .... (2),}$$

$$\text{and } y = -\frac{1}{\mu}(x - \sqrt{a^2 - b^2}) \text{.......... (3),}$$

and the equations corresponding to (4) and (5) will be respectively

$$y = (-)^{\frac{1}{2}}\mu x + \sqrt{a^2\{(-)^{\frac{1}{2}}\mu\}^2 + b^2} \text{ ...... (4),}$$

$$\text{and } y = (-)^{\frac{1}{2}}\frac{1}{\mu}(x - \sqrt{a^2 - b^2}) \text{ .... (5).}$$

And, since by Art. 4, the result of the elimination of μ between the former pair of equations (2), (3), and the latter (4), (5), will be the same, we see clearly, that in this case, as in the parabola, the final equation will represent two loci resulting from the solution of two different problems, exactly analogous to the two problems there discussed.

We will proceed to obtain the final equation.

From (2) we have

$$y - \mu x = (a^2\mu^2 + b^2)^{\frac{1}{2}},$$

and from (3),

$$\mu y + x = (a^2 - b^2)^{\frac{1}{2}}.$$

Squaring these two equations and adding the results, we get

$$(1 + \mu^2)(x^2 + y^2) = (1 + \mu^2)a^2,$$

and therefore

$$(x^2 + y^2 - a^2)(1 + \mu^2) = 0;$$

from which, by the aid of (3), we get

$$(x^2 + y^2 - a^2)\left\{1 + \frac{\{(a^2 - b^2)^{\frac{1}{2}} - x\}^2}{y^2}\right\} = 0,$$

and therefore

$$(x^2+y^2-a^2)\,\{y^2 + [(a^2 - b^2)^{\frac{1}{2}} - x]^2\} = 0.$$

The solution

$$x^2 + y^2 - a^2 = 0$$

belongs to the former problem, and the solution

$$y^2 + \{(a^2 - b^2)^{\frac{1}{2}} - x\}^2 = 0$$

to the latter, being the equations to two straight lines in the plane $+, +^{\frac{1}{4}}$, passing through the focus of the ellipse in the plane

$(+, +)$, and inclined to the axis of x at angles of 45^0, exactly according to the analogy of the lines previously discussed in the case of the parabola.

6. It may not be altogether useless to discuss the algebraical solution of the following problem, for the further elucidation of the principles developed in the preceding disquisitions:

"To determine the locus of the intersections of perpendiculars from the focus of the parabola upon their corresponding tangents, with tangents symmetrically situated on the opposite side of the parabola."

It is evident, on a little reflection, that the equation to the tangent will be

$$y = -ax - \frac{m}{a} \dots\dots\dots\dots (1),$$

and to the intersecting line

$$y = -\frac{1}{a}(x - m) \dots\dots\dots (2).$$

Suppose, now, that $a = -(-)^{\frac{1}{2}}\mu$, and these two equations will evidently become respectively

$$y = (-)^{\frac{1}{2}}\mu x + \frac{m}{(-)^{\frac{1}{2}}\mu} \dots\dots\dots\dots (3),$$

$$\text{and } y = \frac{1}{(-)^{\frac{1}{2}}\mu}(x - m),$$

$$\text{or } y = (-)^{\frac{1}{2}}\frac{-1}{\mu}(x - m) \dots\dots\dots\dots (4);$$

but the equations (3) and (4) are respectively the equations to a tangent to the branch of the parabola in the plane $(+, +^{\frac{1}{4}})$, and to a perpendicular upon it from the focus of the branch in the plane $(+, +)$.

Hence, the equation which comprehends the representation of the former problem will likewise comprehend that of the following:

"To find the equation to the locus of the perpetual intersections of tangents to the branch of the parabola in the plane $(+, +^{\frac{1}{4}})$, with perpendiculars upon them from the focus of the branch in the plane $(+, +)$."

Eliminating between (1) and (2), we get

$$y = \frac{x(x - m)}{y} + \frac{my}{x - m},$$

$$\text{or } y^2(x - m) = x(x - m)^2 + my^2,$$

and therefore $y^2 = \dfrac{x(x-m)^2}{x-2m}$,

which will comprehend both loci.

7. Various additional reflections of a like nature might be made respecting elimination between the general equations of Art. 2. for different affections of a; but what has been said above is sufficient to furnish the correct method of prosecuting such speculations.

W. W.

To the Editor of the Mathematical Journal.

Sir,—Your paper on the Extraneous Solutions of Geometrical Problems, calls my attention to a mistake which I made in an article on the Existence of Branches of Curves in various Planes. I there state (Vol. I., p. 261.) that the extraneous factor of the problem of finding the locus of the intersections of perpendiculars from the focus on the tangents to a parabola, corresponds to the intersection of perpendiculars on the tangents to the branch in the plane perpendicular to xy; whereas it really corresponds to the intersection of those tangents with lines passing through the focus, and making with the axis of x an angle complementary to that of the tangent.

The mistake arose from supposing, that if

$$y = -^{\frac{1}{2}} ax + \beta,$$

in the equation to a tangent in the plane $(+, +^{\frac{1}{4}})$; that to a line perpendicular to it is

$$y = -\frac{1}{-^{\frac{1}{2}} a}(x - m);$$

whereas it ought to be

$$y = - -^{\frac{1}{2}} \frac{1}{a}(x - m);$$

inasmuch as the $-^{\frac{1}{2}}$ has no reference to the mutual inclination of the lines, which is determined only by a. It is true that the locus of the intersections of the tangent with the perpendiculars on it from the focus does pass through the focus, as the geometrical reasoning I have employed shows; but the extraneous factor is not the equation to the locus, which your paper clearly proves. As the nature of the error is, perhaps, not very apparent, I shall feel obliged by your insertion of this correction of it.

I am your obedient servant,

Trin. Coll. D. F. GREGORY.

IX.—ON THE PERSPECTIVE OF THE CO-ORDINATE PLANES.*

I BELIEVE it will be admitted, that very few mathematical writers take any great care to make their diagrams look like what they are intended to represent. This paper will describe a simple and practicable mode of enabling the writer or the student to help himself, his readers, or both, to that clearness of conception which a well-drawn figure never fails materially to aid. The apparatus required will be a card or lamina of wood or ivory, to be presently described, and a small parallel ruler; *no compasses are required*, for ordinary purposes at least.

Let the projection be the orthographic, and let it be given that the projections of the axes make angles as follows:

$$y^{\wedge}z = \alpha, \quad z^{\wedge}x = \beta, \quad x^{\wedge}y = \gamma.$$

The most useful case being that in which the eye is situated (at an infinite distance) in that quarter of space in which all the co-ordinates are positive, I shall assume it as in the figure (see fig. 1, in No. 9, Vol. 2); α, β, and γ must then be severally greater than a right angle. Through OY and the line which (passing through O) is foreshortened into a point, draw a plane: this plane must be perpendicular to ZOX, and must cut it in a line which is projected into the continuation of OY: for as this plane passes through the eye, every angle in it is projected either into nothing or two right angles; the latter in this case, because the eye is, by hypothesis, *in* the angle. Again, any plane which contains the eye, cuts a plane parallel to that of projection in such a manner, that every line which is at right angles to their common intersection is projected into a line making a right angle with the projection of that intersection: if then we draw RP perpendicular to YO produced, we see in RP the projection of a line which is in a parallel plane to that of projection, so that RP is of the same length as its original. Repeat the process with ZO and XO, and we have the following theorem:—Every triangle PQR, which has its vertices on the projected axes, and the origin for the intersection of its *altitudes*, is either a triangle in the plane of projection, or the projection of a triangle parallel to that plane.

From this we shall readily find the projecting cosines of the actual co-ordinates: for the originals of QO, OB, are sides of a right-angled triangle, of which QB is the base, while the invisible perpendicular O′O (O′ is the origin in space) is drawn to its base. Consequently, the ratio of QO′ to its projection QO is the subduplicate ratio of QB to QO; or the co-ordinates x, y, z are foreshortened in the subduplicate ratios of PA to PO, QB to QO, and RC to RO.

* From a Correspondent.

To give an algebraical result, remember that QRP is the supplement of QOP or γ;

$$\text{whence QB} = \text{QZ} \sin \gamma.$$

$$\text{Also, QO} = \text{QZ} \sin \text{QRO} : \sin \text{QOR},$$

$$\text{and} \sin \text{QRO} = \cos \text{RQC} = -\cos \beta;$$

$$\text{while} \sin \text{QOR} = \sin \alpha.$$

Hence the subduplicate ratio of QO : QB is

$$\sqrt{\{-\cos \beta : \sin \alpha \sin \gamma\}};$$

whence the proportions of the projections of a given line on the axes of x, y, and z, are

$$\sqrt{\left(-\frac{\cos \alpha}{\sin \beta \cdot \sin \gamma}\right)}, \quad \sqrt{\left(-\frac{\cos \beta}{\sin \gamma \cdot \sin \alpha}\right)},$$

$$\sqrt{\left(-\frac{\cos \gamma}{\sin \alpha \cdot \sin \beta}\right)} \text{ ...... (Pr. Cos.)},$$

$$\text{or } \sqrt{(-\sin 2\alpha)}, \quad \sqrt{(-\sin 2\beta)}, \quad \sqrt{(-\sin 2\gamma)},$$

which are rational, since 2α, &c., severally lie between two and four right angles.

An algebraical solution might easily be given, but it is hardly worth while. I subjoin, from one before me, the equations necessary for the solution of the inverse problem, namely, Given the projections of equal lines on the three axes, required the angles α, β, γ.

Let p, q, r be the lengths into which l is projected, according as it is parallel to x, y, or z. Then

$$l = \sqrt{\left(\frac{p^2 + q^2 + r^2}{2}\right)},$$

$$\sin \alpha = \frac{l}{bc} \sqrt{(l^2 - a^2)}, \quad \sin \beta = \frac{l}{ca} \sqrt{(l^2 - b^2)}, \quad \sin \gamma = \frac{l}{ab} \sqrt{(l^2 - c^2)}.$$

To apply this to any particular kind of projection, take a card of sufficient size, and having cut out ZOX so that $\angle$ZOX $= \beta$, draw OY so that $\angle$XOY $= \gamma$, and having slit OY, work it with a hard wood or metal point until a pencil can be made to move up and down, guided by OY, upon the paper on which the card is placed. Then calculate $\sqrt{-\sin 2\alpha}$, &c., make scales of equal parts on OX, OY, and OZ, of which the units shall be proportional to these calculated results. These scales may be repeated upon the edges, for convenience of measurement off the axes. The use of these lines is now too evident to need description, and I apprehend, that if the student were in possession of, say, three of them, laid down with different values of α, β, and γ, he would soon find he had better draw his figures for himself, than look at those which are drawn for him in the books.

A. D. M.

X.—THEOREMS IN THE CALCULUS OF GENERATING FUNCTIONS.

To the Editor of the Cambridge Journal.

SIR,—Perhaps some of your readers may take an interest in the following attempts at generalizing some of the theorems of that very graceful weapon of analysis, the Calculus of Generating Functions.

(1) Let $\phi(t)$ represent the generating function of u_x or $\phi(t) = G(u_x)$.

Then, if $\psi(t)$ be any function of t capable of being expanded in integral powers of t, positive or negative, $\phi(t).\psi(t)$ will be the generating function of $\psi\left(\frac{1}{1+\Delta}\right)u_x$, or of $\psi\left(\frac{1}{D}\right)u_x$, if $1+\Delta = D$.

$$\text{For let } \psi(t) = \ldots + \frac{a_{-1}}{t} + a_0 + a_1 t + a_2 t^2 + \ldots$$

$$\text{then since } \phi(t) = u_0 + u_1 t + u_2 t^2 + \ldots\ldots\ldots$$

we have for the coefficient of t^x in the product of these two series,

$$\ldots\ldots a_{-1}u_{x+1} + a_0 u_x + a_1 u_{x-1} + \ldots\ldots$$

$$= (\ldots a_{-1}D + a_0 + \frac{a_1}{D} + \ldots)\, u_x = \psi\left(\frac{1}{D}\right) u_x,$$

$$\text{i. e. } \phi(t)\,\psi(t) = G\,.\,\psi\left(\frac{1}{D}\right) u_x.$$

Similarly, we may shew that

$$\phi(t)\,\psi\left(\frac{1}{t}\right) = G\psi(D)\,u_x.$$

From these theorems we may demonstrate two problems in the Senate-House Papers for January 1840.

For, from what has been said, it follows immediately that

$$(a-\log t)^{-1}\phi(t) = \left\{a + \log\left(\frac{1}{t}\right)\right\}^{-1}\phi(t) = G\,.\,(a+\log D)^{-1}u_x$$

$$= G\,.\left(\frac{d}{dx} + a\right)^{-1} u_x, \ (\text{since } D = \varepsilon^{\frac{d}{dx}}),$$

$$= G\,\varepsilon^{-ax}\int \varepsilon^{ax} u_x\,dx. \quad (\text{See Vol. I. No. I.})$$

This example, combined with the preceding theorem, renders it unnecessary to state that

$$\phi(t)\,\psi(\log t) = G\psi\left\{\log\left(\frac{1}{D}\right)\right\}.\,u_x,$$

$$\text{and } \phi(t)\,\psi\left(\log\frac{1}{t}\right) = G\psi\left(\frac{d}{dx}\right).\,u_x,$$

$$\phi(t)\,.\,\log^n\frac{1}{t} = G\,\frac{d^n u_x}{dx^n}.$$

Again, for the second problem we have

$$(1 - at)^{-1}\, t\phi(t) = \mathrm{G}\,.\,(\mathrm{D} - a)^{-1}\, u_x = \mathrm{G}\,.\,a^{x-1}\, \varepsilon\, \frac{u_x}{a_x}.$$

(See Vol. I. No. II.)

Thus, to expand $(a + \log t)^{-2} \dfrac{t}{(1-t)^2}$, we may write it under the form

$$\left(a - \log \frac{1}{t}\right)^{-2} \frac{t}{(1-t)^2} = f(t), \text{ suppose.}$$

$$\text{Now, } \frac{t}{(1-t)^2} = \mathrm{G}\,.\,x;$$

$$\therefore f(t) = \mathrm{G}\left(a - \frac{d}{dx}\right)^{-2}.\,x$$

$$= \mathrm{G}\left\{\frac{1}{a^2}\left(1 + \frac{2}{a}\right) + (c_1 + c_2 x)\right\}\varepsilon^{ax}$$

$$= \frac{1}{a^2} + c_1 + \left\{\frac{1}{a^2}\left(1 + \frac{2}{a}\right) + (c_1 + c_2)\,\varepsilon^{a}\right\} t$$

$$+ \left\{\frac{1}{a^2}\left(1 + \frac{2}{a}\right) + (c_1 + 2c_2)\,\varepsilon^{2a}\right\} t^2 + \ldots$$

To determine the constants, put $t = 0$;

$$\therefore c_1 = -\frac{1}{a^2}.$$

Divide both sides by t, and make t again $= 0$, the value of the first side will still be 0;

$$\therefore \frac{1}{a^2}\left(1 + \frac{2}{a}\right) + \left(c_2 - \frac{1}{a^2}\right)\varepsilon^{a} = 0;$$

$$\therefore c_2 = \frac{1}{a^2}\left\{1 - \left(1 + \frac{2}{a}\right)\varepsilon^{-a}\right\}.$$

From these theorems we may derive the following methods of solving linear equations, either differential or of differences.

Thus, let $\psi(\mathrm{D})\, u_x = \mathrm{A}_x$ be the equation of differences. Then, if we can find $\mathrm{F}(t)$, the generating function of A_x, (which may frequently be done without difficulty,) we have, equating the generating functions on both sides,

$$\psi\left(\frac{1}{t}\right)\phi(t) = \mathrm{F}(t)$$

$$= \phi(t) = \frac{\mathrm{F}(t)}{\psi\left(\frac{1}{t}\right)},$$

and $v_x =$ coefficient of t^x in the expansion of $\dfrac{\mathrm{F}(t)}{\psi\left(\frac{1}{t}\right)}$.

The same method applies, *mutatis mutandis,* to the differential equations, but it is in both cases difficult of application, as the coefficient of t^x in $\dfrac{F(t)}{\psi\left(\frac{1}{t}\right)}$ can only be obtained by the enunciation of a series whose formation is generally complicated.

E. J. L.

XI.—MATHEMATICAL NOTE.

The following is an easy method of finding the number of homogeneous products of n dimensions of m letters:—

It is evident that the number of such products is the same as the number of the terms of the n^{th} degree in the product

$$(1 + x_1 + x_1^2 + \ldots + x_1^n)(1 + x_2 + x_2^2 + \ldots + x_2^n) \ldots$$
$$\ldots (1 + x_m + x_m^2 + \ldots + x_m^n);$$

and this number is clearly equal to the coefficient of the n^{th} power of x in the development of

$$(1 + x + x^2 + \ldots + x^n)^m,$$

or, which is the same thing, in the development of

$$(1 + x + x^2 + \ldots \text{ to } \infty)^m,$$
$$\text{or of } (1 - x)^{-m},$$

which is

$$\frac{m(m+1) \ldots\ldots (m+n-1)}{1 \,.\, 2 \,.\, 3 \ldots\ldots n}.$$

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CORRIGENDA.

In Vol. I. p. 213, line 12, should be written

$$d^n u = \frac{d^n u}{dx^n} dx^n + n \frac{d^n u}{dx^{n-1}dy} dx^{n-1}dy + \frac{n(n-1)}{n} \frac{d^n u}{dx^{n-2}dy^2} dx^{n-2}dy^2$$
$$+ \&c.$$

In Vol. II. p. 21, line 15, *read* 'a stratum' *instead of* 'attraction.'